Topology I and II Notes

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Contents

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1 Topology I

1.12 Section 12: Topological Spaces

Definitions: topology, open set, closed set, trivial topology, discrete topology, finite complement topology, finer, coarser
Main Idea: Introduce the idea of a topology on a set.

Definition 1.12.1. A **topology** τ on a set X is a collection of subsets of X such that:

1. $\emptyset$ and X are in τ,
2. τ is closed under arbitrary unions; i.e., if $\{U_\alpha\}_{\alpha \in J}$ is an indexed subcollection of τ, then $U = \bigcup_{\alpha \in J} U_\alpha$ is in τ, and
3. τ is closed under finite intersections; i.e., if $\{U_i\}_{i=1}^n$ is a finite subcollection of τ, then $U = \bigcap_{i=1}^n U_i$ is in τ.

We write (X, τ) to refer to the set X with topology τ. Together they are a topological space. If the topology is obvious, we often just write X.

Definition 1.12.2. If $U \subseteq X$ is in τ, we call U a **open set**.

Definition 1.12.3. Let $A \subseteq X$. We say A is a **closed set** if $X \setminus A$ is open.

Definition 1.12.4. $(X, \{\emptyset, X\})$, the smallest possible topology on X, is called the **trivial topology**. The only open sets are $\emptyset$ and X itself.

Definition 1.12.5. $(X, \mathcal{P}(X))$, the largest possible topology of X, is called the **discrete topology**. Every subset of X is an open set.

Definition 1.12.6. Let $\tau_f = \{U \subseteq X \mid U = \emptyset \text{ or } |X \setminus U| < \infty\}$. (X, τ_f) is the **finite complement topology**.

We can compare two topologies on a set X:

Definition 1.12.7. If τ and τ' are topologies on X, τ' is **finer** than τ if $\tau \subseteq \tau'$. We also say τ is **coarser** than τ'.

The gravel analogy doesn't sit well with me yet: Let the gravel be the open sets of τ. We smash the gravel into smaller chunks, creating τ', and τ' is finer than τ. It's easier for me to think τ is <u>c</u>oarser than τ' if $\tau \subseteq \tau'$.

Note also that two topologies on X need not be comparable.

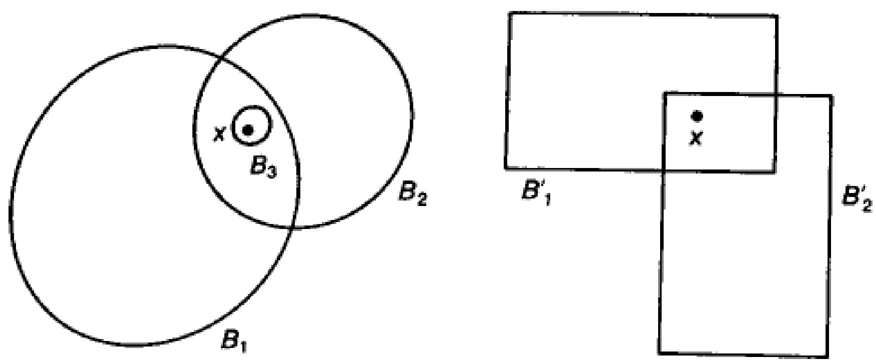
1.13 Section 13: Basis for a Topology

Definitions: basis, standard topology, lower limit topology, K-topology
Main Idea: It's actually rather hard to list every open set in τ. A basis lets us describe the topology anyways.

Definition 1.13.1. A collection of subsets $\mathcal{B}$ of a set X is a **basis** for a topology on X if:

1. $X = \bigcup_{B \in \mathcal{B}} B$; i.e., for all $x \in X$, there exists $B \in \mathcal{B}$ such that $x \in B$, and
2. if $x \in B_1 \cap B_2$, then there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

Condition 2. can be illustrated succinctly:



Theorem 1.13.2. *If $\mathcal{B}$ is a basis, then the topology τ generated by $\mathcal{B}$ is the collection of $U \subseteq X$ such that for all $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$.*

Proof. We show that τ is a topology.

- $\emptyset \in \tau$, vacuously.
- $X \in \tau$, as $\mathcal{B}$ is a basis exactly when $X = \bigcup_{B \in \mathcal{B}} B$, so for all $x \in X$, there exists $B \in \mathcal{B}$ such that $x \in B$ which is naturally a subset of X.
- Closed under unions: Suppose $\{U_\alpha\}_{\alpha \in J} \in \tau$ and let $U = \bigcup_\alpha U_\alpha$. If $x \in U$, then $x \in U_\alpha$ for some α. Since $U_\alpha \in \tau$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U_\alpha \subseteq U$, so $U \in \tau$.
- Closed under finite intersections: We give the structure of an inductive proof: Let $U_1, U_2 \in \tau$. If $x \in U_1 \cap U_2$, then there exist $B_1, B_2 \in \mathcal{B}$ such that $x \in B_1 \subseteq U_1$ and $x \in B_2 \subseteq U_2$. Then $x \in B_1 \cap B_2$, so there exists

$B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2 \subseteq U_1 \cap U_2$. Thus $U_1 \cap U_2 \in \tau$. By induction, if $U_1, ..., U_n \in \tau$, so is $\bigcap_{i=1}^{n} U_i$. $\square$

But it turns out there are other ways to characterize the topology τ generated by the basis $\mathcal{B}$.

Lemma 1.13.3. *Let X be a set. Let $\mathcal{B}$ be a basis for a topology τ on X. Then $U \in \tau$ if and only if $U = \bigcup_{\alpha} B_\alpha$ for some collection $\{B_\alpha\} \subseteq \mathcal{B}$.*

Proof. If $U = \bigcup_{\alpha} B_\alpha$, well, since $B_\alpha \in \mathcal{B} \subseteq \tau$ and τ is closed under union, then $U \in \tau$.

Conversely, if $U \in \tau$, then for all $x \in U$, there exists $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq U$ (**Theorem 1.13.2**). Well, that means $U = \bigcup_{x \in U} B_x$ for the collection $\{B_x\} \subseteq \mathcal{B}$. $\square$

Note that the basis for a topology is **not** unique! We had tons of choices in the lemma above.

What about finding a basis, given a topology? Here is one way:

Lemma 1.13.4. *Suppose (X, τ) is a topological space and $\mathcal{C} \subseteq \tau$ for which:*

for all $U \in \tau$, if $x \in U$, then there exists $C \in \mathcal{C}$ such that $x \in C \subseteq U$.

Then $\mathcal{C}$ is a basis, generating the topology τ'. Furthermore, $\tau' = \tau$.

Proof. We first show that $\mathcal{C}$ is a basis; recall **Definition 1.13.1**.

- $X = \bigcup_{C \in \mathcal{C}} C$? Yes. As $X \in \tau$, if $x \in X$, then there exists $C_x \in \mathcal{C}$ such that $x \in C_x \subseteq X$. Therefore $X = \bigcup_{x} C_x$.
- If $x \in B_1 \cap B_2$, there exists B_3 such that $x \in B_3 \subseteq B_1 \cap B_2$? Yes. Let $C_1, C_2 \in \mathcal{C} \subseteq \tau$. Then $C_1 \cap C_2 \in \tau$. Thus there exists $C_3 \in \mathcal{C}$ such that $x \in C_3 \subseteq C_1 \cap C_2$. Therefore, $\mathcal{C}$ is a basis, generating τ'. Let us now show $\tau' = \tau$.

We see that $\tau \subseteq \tau'$: Let $U \in \tau$. Then there exists $C \in \mathcal{C} \subseteq \tau'$ such that $x \in C \subseteq U$, so $U \in \tau'$, by **Theorem 1.13.2**.

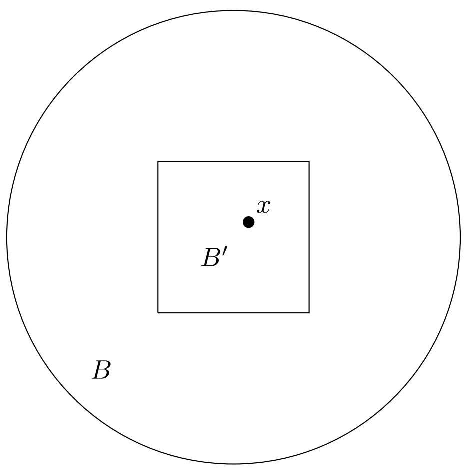
Now see that $\tau' \subseteq \tau$: Let $U \in \tau'$. By **Lemma 1.13.3**, $U = \bigcup_{\alpha} C_\alpha$ for $\{C_\alpha\} \subseteq \mathcal{C} \subseteq \tau$, so $U \in \tau$. Therefore, $\tau' = \tau$. $\square$

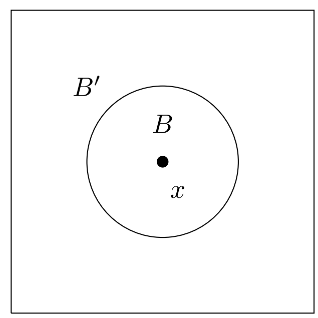
Lemma 1.13.5. *Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for τ and τ' on X. Then $\tau \subseteq \tau'$ if and only if for all $x \in X$ and $B \in \mathcal{B}$ such that $x \in B$, there exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.*

Proof. First, assume $\tau \subseteq \tau'$. Fix an $x \in X$ and $B \in \mathcal{B}$ where $x \in B$. Then as $\mathcal{B} \subseteq \tau \subseteq \tau'$, B is open in τ'. Since τ' is generated by $\mathcal{B}'$, there exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$ (**Theorem 1.13.2**), as desired.

Conversely, assume for all $x \in X$ and $B \in \mathcal{B}$ with $x \in B$, there exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$. Let $U \in \tau$ and $x \in U$. Then **Theorem 1.13.2** gives that there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$. Thus there exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B \subseteq U$, so $U \in \tau'$. Therefore, $\tau \subseteq \tau'$. $\square$

Lemma 1.13.5 can show that the Euclidean ball topology and the square metric topology, B_d and B_ρ, are the same. Compare each of their bases; show each is finer than the other.





Definition 1.13.6. Let $\mathcal{B} = \{(a,b) \mid a,b \in \mathbf{R}, a < b\}$. The topology generated by $\mathcal{B}$ is called the **standard topology**, and without clarification, this is the default topology on $\mathbf{R}$.

Definition 1.13.7. The topology generated by $\mathcal{B}' = \{[a,b)\}$ is the **lower limit topology** on $\mathbf{R}$, denoted $\mathbf{R}_l$ for $(\mathbf{R}, \tau')$.

Definition 1.13.8. Let $K = \left\{\frac{1}{n} \mid n \in \mathbf{N}\right\}$. Let $\mathcal{B}'' = \{(a,b)\} \cup \{(a,b) \setminus K\}$. The topology generated by $\mathcal{B}''$ is called the K**-topology**, and $(\mathbf{R}, \tau'')$ is written $\mathbf{R}_K$.

Lemma 1.13.9. *The topologies of $\boldsymbol{R}_l$ and $\boldsymbol{R}_K$ are strictly finer than the standard topology on $\boldsymbol{R}$, but are not comparable with each other.*

Proof. Let τ, τ', and τ'' be the topologies of $\mathbf{R}$, $\mathbf{R}_l$, and $\mathbf{R}_K$, with bases $\mathcal{B}$, $\mathcal{B}'$, and $\mathcal{B}''$. We use **Lemma 1.13.5** to compare topologies.

Let $x \in (a,b) \in \mathcal{B} \subseteq \tau$. Then $x \in [x,b) \subseteq (a,b)$. And if $[x,d) \in \mathcal{B}' \subseteq \tau'$, there does not exist $(a,b) \in \mathcal{B}$ such that $x \in (a,b) \subseteq [x,d)$. Therefore $\tau \subseteq \tau'$.

Now we compare $\mathbf{R}$ and $\mathbf{R}_K$. Let $x \in (a,b) \in \mathcal{B} \subseteq \tau$; (a,b) is also a basis element of τ'' containing x. However, in the other direction, if $B'' = (-1,1)\backslash K \in \mathcal{B}''$, $0 \in B''$ but no open interval (a,b) exists such that $0 \in (a,b) \subseteq B''$. Therefore $\tau \subseteq \tau''$.

Now we show τ' and τ'' are not comparable, i.e., $\tau' \not\subseteq \tau''$ abd $\tau'' \not\subseteq \tau'$.

For $\tau' \not\subseteq \tau''$, let $[-1,0) \in \tau'$. Then no $B'' \in \tau''$ satisifies $-1 \in B'' \subseteq [-1,0)$.

For $\tau'' \not\subseteq \tau'$, let $(-1,1) \setminus K \in \tau''$. Then no $B' \in \tau'$ satisfies $0 \in B' \subseteq (-1,1) \setminus K$. $\square$

1.14 Section 14: The Order Topology

Definitions: order topology
Main Idea: Let's explore this specific topology.

Suppose X is a simply ordered set; we can define intervals in the usual way: $(a,b) = \{x \in X \mid a < x < b\}$. We can define half open and closed intervals as well.

Proposition 1.14.1. *Let X be a simply ordered set and let $\mathcal{B}$ be the collection of subsets of X where at least one holds for each $B \in \mathcal{B}$:*

1. *$B = (a,b)$,*
2. *$B = [a_0, b)$ if a_0 is the least element (if any) of X, or*
3. *$B = (a, b_0]$ if b_0 is the greatest element (if any) of X.*

*Then $\mathcal{B}$ is a basis and we call the topology generated by $\mathcal{B}$ the **order topology**.*

Note that if $X = \mathbf{R}$, then the order topology is the standard topology.

Example 1.14.2. Let $X = \{1,2\} \times \mathbf{N}$. We can order X lexicographically (dictionary order): $(a_0,b_0) \leq (a_1,b_1)$ if $a_0 < a_1$, or $a_0 = a_1$ and $b_0 \leq b_1$.

We can see that this topology is not the discrete topology, because $\{(2,1)\}$ is not open. To see this, see that the ordering is

$$(1,1),(1,2),(1,3),(1,4),... \qquad\qquad (2,1),(2,2),(2,3),...$$

Any open interval containing the point $(2,1)$ contains $(1,n)$ for large n, so $\{(2,1)\}$ is not open, and hence this is not the discrete topology, where every set is open.

1.15 Section 15: The Product Topology

Definitions: product topology, projection
Main Idea: We define the product topology in terms of its basis. We'll define it explicitly later.

Definition 1.15.1. Let X and Y be topological spaces. The **product topology** on $X \times Y$ is the topology generated by the basis $\mathcal{B} = \{U \times V \mid U \subseteq X \text{ is open and } V \subseteq Y \text{ is open.}\}$.

Let's confirm that $\mathcal{B}$ is a basis (recall **Definition 1.13.1**):

1. As $X \subseteq X$ is open, $Y \subseteq Y$ is open, we have $X \times Y \in \mathcal{B}$, so $X \times Y$ is a union of sets in $\mathcal{B}$.

2. If $U_1 \times V_1$ and $U_2 \times V_2 \in \mathcal{B}$, then so is $(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2)$.

Thus $\mathcal{B}$ is a basis. Note that $\mathcal{B}$ is **not** a topology though. It is not closed under unions; the union of two rectangles in $X \times Y$ is almost never a rectangle, and hence not in $\mathcal{B}$. But it is open.

Example 1.15.2. The product topology on $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$ gives the standard topology on $\mathbf{R}^2$.

Note that the topology of the space $X \times Y$ is generated by the basis of products of basis elements of X and Y. So for the standrard topology in $\mathbf{R}^2$, the basis consists of rectangles $(a, b) \times (c, d)$.

Definition 1.15.3. A **projection** is a map $\pi_i : X_1 \times X_2 \to X_i$ such that $\pi_i(x_1, x_2) = x_i$ for $i = 1, 2$. We can actually generalize projection to any number of factors.

1.16 Section 16: The Subspace Topology

Definitions: subspace topology
Main Idea: Explore the subspace topology and prove a few ideas.

Definition 1.16.1. Suppose X is a topological space and $Y \subseteq X$. Then the **subspace topology** on Y is the collection $\tau_Y = \{U \cap Y \mid U \subseteq X \text{ is open.}\}$.

Lemma 1.16.2. *If $\mathcal{B}$ is a basis for a topology on X, then $\mathcal{B}_Y = \{B \cap Y \mid B \in \mathcal{B}\}$ is a basis for the subspace topology on Y.*

Proof. Suppose $U \subseteq X$ is open, and $y \in U \cap Y$. As $y \in U \subseteq X$, there exists $B \in \mathcal{B}$ such that $y \in B \subseteq U$. Thus $y \in B \cap Y \subseteq U \cap Y$. □

Example 1.16.3. If $Y = [0, 1] \subseteq \mathbf{R}$, a basis for the subspace topology then consists of $Y \cap (a, b)$:

- (a, b) if $a, b \in Y$,
- $[0, b)$ if $a \notin Y$, $b \in Y$,
- $(a, 1]$ if $a \in Y$, $b \notin Y$,

- $\emptyset$ or $[0, 1]$ if $a, b \notin Y$.

Example 1.16.4. Let $Y = [0, 1] \cup \{2\} \subseteq \mathbf{R}$. Then $\{2\}$ is open in the subspace topology as $\{2\} = \left(\frac{3}{2}, \frac{5}{2}\right) \cap Y$.

So you can have sets open in Y that are not open in $\mathbf{R}$; i.e., $[0, 1]$ or $\{2\}$. BUT

Lemma 1.16.5. *Let Y be a subspace of X. If U is open in Y, and Y is open in X, then U is open in X.*

Proof. If U is open in Y, then $U = V \cap Y$ for some $V \subseteq X$ where V is open. So if Y is open in X, then $U = V \cap Y$ must be open in X. $\square$

Next, we prove a claim we alluded to in passing in the previous section:

Theorem 1.16.6. *Let $A \subseteq X$ and $B \subseteq Y$. The product topology on $A \times B$, call it τ, is equal to the subspace topology τ' with $A \times B \subseteq X \times Y$.*

Proof. By definition, a basis for τ consists of $\{(U \cap A) \times (V \cap B) \mid U \subseteq X \text{ open}, V \subseteq Y \text{ open}\}$. And a basis for τ' consists of $\{(U \times V) \cap (A \times B) \mid U \subseteq X \text{ open}, V \subseteq Y \text{ open}\}$. Since $(U \cap A) \times (V \cap B) = (U \times V) \cap (A \times B)$, these bases are equal. $\square$

1.17 Section 17: Closed Sets and Limit Points

Definitions: interior, closure, limit point, convergence, Hausdorff
Main Idea: Interiors and closures are what you'd expect. Limit points as well. Convergence is screwy, but Hausdorff spaces mitigate that.

Recall **Definition 1.12.3**: a set $C \subseteq X$ is closed if $X \setminus C$ is open.

Theorem 1.17.1. *Let $Y \subseteq X$ be a subspace. A set $A \subseteq Y$ is closed in Y if and only if $A = C \cap Y$ for some closed set $C \subseteq X$.*

Proof. First, if $C \subseteq X$ is closed, then $X \setminus C$ is open, so $Y \cap (X \setminus C)$ is open in Y by **Definition 1.16.1**. Hence $A = C \cap Y = Y \setminus (Y \cap (X \setminus C))$ is closed in Y.

Next, if A is closed in Y, then $A = Y \setminus U$ for some U open in Y. Then $U = Y \cap V$ for some $V \subseteq X$ open, and hence $A = Y \setminus (Y \cap V) = Y \cap (X \setminus V)$, where $X \setminus V$ is closed in X. $\square$

Definition 1.17.2. Let $A \subseteq X$. The **interior** of A, $\text{Int}(A) \subseteq A$, is the largest open set inside of A, i.e., the union of all open sets in A.

Definition 1.17.3. Let $A \subseteq X$. The **closure** of A, $A \subseteq \overline{A}$, is the smallest closed set containing A, i.e., the intersection of all closed sets containing A.

Lemma 1.17.4. *Let Y be a subspace of X and let $A \subseteq Y$. The closure of A in Y equals the closure of A in X, intersect Y.*

Proof. Denote the closure of A in X by $\overline{A}$. Let B be the closure of A in Y. Then since $\overline{A}$ is closed, $\overline{A} \cap Y$ is closed in Y. Since $A \subseteq \overline{A} \cap Y$, we have $B \subseteq \overline{A} \cap Y$.

Conversely, $B = Y \cap C$ for some closed set $C \subseteq X$. Thus $A \subseteq B \subseteq C$, hence $\overline{A} \subseteq C$. Therefore $\overline{A} \cap Y \subseteq C \cap Y = B$. $\square$

Theorem 1.17.5. *Let $A \subseteq X$. Then $x \in \overline{A}$ if and only if for all open sets $U \subseteq X$ with $x \in U$, $U \cap A \neq \emptyset$. We say that U is a* ***neighborhood*** *of x.*

Proof. If $x \notin \overline{A}$, then $U = X \setminus \overline{A}$ is a neighborhood of x disjoint from A.

Conversely, if U is a neighborhood of x disjoint from A, i.e., $U \cap A = \emptyset$, then $X \setminus U$ is a closed set containing A. Hence $\overline{A} \subseteq X \setminus U$. But since $x \in U$, $x \notin X \setminus U$, so $x \notin \overline{A}$. $\square$

Let's see some examples of closure:

Example 1.17.6. Let $Y = (0, 1] \subseteq \mathbf{R}$ with the standard topology. Then $\overline{Y} = [0, 1]$.

Example 1.17.7. Let $Y = \{1\} \times \mathbf{N} \subseteq \{1, 2\} \times \mathbf{N}$ with the order topology. Then $\overline{Y} = Y \cup \{(2, 1)\}$.

Definition 1.17.8. If X is a topological space and $A \subseteq X$, we say that $x \in X$ is a **limit point** of A if every neighborhood of x contains some point in $A \setminus \{x\}$. Therefore, $x \in \overline{A \setminus \{x\}}$.

Note that x may or may not lie in A. We let A' be the set of limit points of A.

Example 1.17.9. Let $Y = (0, 1] \subseteq \mathbf{R}$ with the standard topology. Then $Y' = [0, 1]$.

Example 1.17.10. Let $Y = \{1\} \times \mathbf{N} \subseteq \{1, 2\} \times \mathbf{N}$ with the order topology. Then $Y' = \{(2, 1)\}$.

These examples lead us to the following theorem:

Theorem 1.17.11. *$\overline{A} = A \cup A'$.*

Proof. First, let $x \in A'$. Then $x \in \overline{A \setminus \{x\}} \subseteq \overline{A}$. And by definition $A \subseteq \overline{A}$, so $A \cup A' \subseteq \overline{A}$.

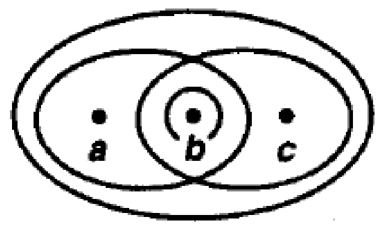
Conversely, let $x \in \overline{A}$. If $x \in A$, we are done. Suppose $x \notin A$. Then every neighborhood of x intersects $A = A \setminus \{x\}$, and hence $x \in A'$. Thus, $\overline{A} \subseteq A \cup A'$. $\square$

Corollary 1.17.12. *A is closed if and only if $A' \subseteq A$.*

Definition 1.17.13. A sequence $x_1, x_2, \ldots$ in X **converges** to $x \in X$ if for all neighborhoods U of x, there exists N such that if $n \geq N$ then $x_n \in U$.

Be careful with your intuition, however!

Example 1.17.14. Let $X = \{a, b, c\}$ with the topology described in the picture adjacent. Then there are two problems. The first is that c and a are limit points of $\{b\}$, since every nonempty open set contains c. The second problem is that the constant sequence $c, c, c, ...$ converges to a, b, and c!



To help fix this, we introduce the following definition:

Definition 1.17.15. A topological space X is **Hausdorff** if for all $x \neq y \in X$ there exist neighborhoods U and V of x and y respectively such that $U \cap V = \emptyset$. This is also called the T_2 separation axiom.

Shakil's quip is helpful here: a space is Hausdorff if two points can be "housed off" from each other via open sets.

Theorem 1.17.16. *In a Hausdorff space, finite sets are closed.*

Proof. Let X be a Hausdorff space. We show that singletons $\{x\} \in X$ are closed, and then a finite union of singletons is still finite and is still closed. Let $x \neq y \in X$. Then as X is Hausdorff, there exist U and V, disjoint neighborhoods of x and y, respectively. As $U \cap \{y\} = \emptyset$, $y \notin \overline{\{x\}}$. Hence $\overline{\{x\}} = \{x\}$, so $\{x\}$ is closed. □

Corollary 1.17.17. *The only Hausdorff topology on a finite set is the discrete topology.*

Recall that we introduced the definition of Hausdorff to deal with unintuitive convergence behavior. We prove our claim here:

Theorem 1.17.18. *If X is a Hausdorff space, then a sequence of points converges to at most one point.*

Proof. Suppose that $x_n \to x$. Let $y \neq x$; we show that $x_n \not\to y$. Let U and V be disjoint neighborhoods of x and y. Since $x_n \to x$, all $x_n \in U$ for large enough n. Since U and V are disjoint, these $x_n \notin V$, so $x_n \not\to y$. □

Example 1.17.19. Suppose we have a sequence $(x_n) \subseteq \mathbf{R}$ bounded and monotone. If (x_n) is increasing, we have convergence in the standard topology (of course), and also in the upper limit topology, but not in the lower limit topology. Likewise, if (x_n) is decreasing, (x_n) converges in the lower limit topology, but not the upper limit topology.

In the K-topology, convergent sequences converge, unless they are eventually subsequences of K.

1.18 Section 18: Continuous Functions

Definitions: preimage, continuous, homeomorphism, open map
Main Idea: In a continuous function, preimages of open sets are open. A homeomorphism is a continuous bijection with a continuous inverse.

Given a function $f : X \to Y$, we can map specific pieces of X to Y, as there exists a function $f : \mathcal{P}(X) \to \mathcal{P}(Y)$. This is just to say a function from subsets of X to subsets of Y makes sense. We call the image of $A \subseteq X$ $f(A) = \{f(a) \mid a \in A\}$.

We'd like to see how functions map topologies, but there's a problem: functions don't behave well with respect to set operations:

- $f(A \cap B) \neq f(A) \cap f(B)$ (although $f(A \cap B) \subseteq f(A) \cap f(B)$), and
- $f(X \setminus A) \neq Y \setminus f(A)$.

So we can't just consider functions that take open sets to open sets.

The idea is that we use preimages instead:

Definition 1.18.1. Consider $f^{-1} : \mathcal{P}(Y) \to \mathcal{P}(X)$. The **preimage** of $B \subseteq Y$ is $f^{-1}(B) = \{x \in X \mid f(x) \in B\}$.

This does respect set operations:

- $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$,
- $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$, and
- $X \setminus f^{-1}(A) = f^{-1}(Y \setminus A)$.

Definition 1.18.2. Let (X, τ_X) and (Y, τ_Y) be topological spaces. A function $f : X \to Y$ is **continuous** if $f^{-1}(\tau_Y) \subseteq \tau_X$. In other words, preimages of open sets in Y are open in X.

Note that if τ_Y is generated by a basis $\mathcal{B}_Y$, then it suffices to have that $f^{-1}(\mathcal{B}_Y) \subseteq \tau_X$.

Theorem 1.18.3. *Let $f : X \to Y$. Then the following are equivalent:*

1. *f is continuous.*
2. *For all subsets $A \subseteq X$, $f(\overline{A}) \subseteq \overline{f(A)}$.*
3. *For all closed subsets $B \subseteq Y$, $f^{-1}(B) \subseteq X$ is closed.*
4. *For all $x \in X$ and neighborhoods $V \subseteq Y$ of $f(x)$, there exists a neighborhood U of x such that $f(U) \subseteq V$.*

Proof. For 1. yields 2., assume that f is continuous. Let $A \subseteq X$, and let $x \in \overline{A}$. We wish to show $f(x) \in \overline{f(A)}$. Let V be a neighborhood of $f(x)$. Then $f^{-1}(V)$ is an open set of X containing x, so it must intersect A at some point y, by **Theorem 1.17.5**. Then V intersects $f(A)$ at $f(y)$, so $f(x) \in \overline{f(A)}$, as desired.

For 2. yields 3., let $B \subseteq Y$ be closed and let $A = f^{-1}(B)$. We wish to show that $A \subseteq X$ is closed, so we show that $\overline{A} = A$. We know that $f(A) = f(f^{-1}(B)) \subseteq B$. Let $x \in \overline{A}$, and then $f(x) \inf(\overline{A}) \subseteq \overline{f(A)} \subseteq \overline{B} = b$, so $x \in f^{-1}(B) = A$. Thus $\overline{A} \subseteq A$, so $\overline{A} = A$, as desired.

For 3. yields 1., let $V \subseteq Y$ be open. Let $B = Y \setminus V$, closed. Then $f^{-1}(B) = f^{-1}(Y \setminus V) = f^{-1}(Y) \setminus f^{-1}(V) = X \setminus f^{-1}(V)$. Since $f^{-1}(B)$ is closed in X, $f^{-1}(V)$ is open in X, as desired.

For 1. yields 4., let $x \in X$ and let V be a neighborhood of $f(x)$. Then $U = f^{-1}(V)$ is a neighborhood of x such that $f(U) \subseteq V$, as desired.

For 4. yields 1., let $V \subseteq Y$ be open. Let $x \in f^{-1}(V)$. Then $f(x) \in V$, so by hypothesis there exists a neighborhood U_x of x such that $f(U_x) \subseteq V$. Then $U_x \subseteq f^{-1}(V)$, so $f^{-1}(V) = \bigcup_x U_x$, so $f^{-1}(V)$ is open, as desired. □

Definition 1.18.4. If $f : X \to Y$ is a continuous bijection and $f^{-1} : Y \to X$ is continuous, then f is a **homeomorphism**.

Note that since f being a continuous bijection implies f^{-1} is a bijection, we see that f^{-1} is also a homeomorphism. Furthermore, since $(f^{-1})^{-1} = f$, the map f^{-1} is continuous if and only if, if $U \subseteq X$ is open, then $f(U) = (f^{-1})^{-1}(U) \subseteq Y$.

Definition 1.18.5. The map $f : X \to Y$ is called a **open map** if for any set $U \subseteq X$ open, the image $f(U) \subseteq Y$ is open.

So f^{-1} is continuous if and only if f is an open map. Therefore a continuous open bijection is a homeomorphism.

Example 1.18.6. We show that $\mathbf{R}$ is homeomorphic to $(-1, 1)$ via an explicit homeomorphism. Let $f(x) = \frac{x}{|x|+1}$. Then f is continuous and a bijection, and $f^{-1}(y) = \frac{y}{1-|y|}$ is continuous.

Example 1.18.7. Eric would prefer the homeomorphism $g : \mathbf{R} \to (-1, 1)$, $g(x) = \frac{2}{\pi} \arctan(x)$.

Example 1.18.8. Be aware that not every continuous bijection is a homeomorphism! Of course, just by the definition, we can find an example where f^{-1} is not continuous. Consider $X = [0, 1) \subseteq \mathbf{R}$ and $S^1 = \{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 = 1\} \subseteq \mathbf{R}^2$. Then let $f(t) = (\cos(2\pi t), \sin(2\pi t))$. Then $f : X \to S^1$ is a continuous bijection, but $U = \left[0, \frac{1}{4}\right) \subseteq X$ is open and $f(U)$ is not open in S^1.

Now we list some properties of continuous functions:

1. Constant functions are continuous. To see this, take $V \subseteq Y$ open. Then if V hits the image of f, $f^{-1}(V) = X$ open, and if V doesn't hit the image of f, $f^{-1}(V) = \emptyset$ also open.

2. The inclusion of a subspace is continuous. To see this, let the inclusion be $i : A \hookrightarrow X$. Then if V is open in X, $f^{-1}(V) = V \cap A$, open in A.

3. Compositions of continuous functions are continuous. To see this, let $X \xrightarrow{f} Y \xrightarrow{g} Z$ and let $W \subseteq Z$ open. Then as g is continuous, $g^{-1}(W)$ is open, and as f is continuous, $f^{-1}(g^{-1}(W))$ is open, as desired.

4. If $f : X \to Y$ is continuous and $A \subseteq X$, then $f|_A : A \to Y$ is continuous. To see this, see that $f|_A = fi$. Then use properties 2. and 3.

5. If $Y \subseteq Z$ and $f : X \to Y$ is continuous, then so is $f : X \to Z$. To see this, if $V \subseteq Y$ is open, then $V \subseteq Z$ is open.

6. If $f : X \to Y$ is continuous, then $f : X \to f(X) \subseteq Y$ is continuous. Everything outside of $f(X)$ isn't seen by f.

7. If $X = \bigcup\limits_{\alpha} U_\alpha$ and $f|_{U_\alpha} : U_\alpha \to Y$ is continuous, then $f : X \to Y$ is continuous. To see this, notice that for $V \subseteq Y$ open, $f^{-1}(V) \cap U_\alpha = \left(f|_{U_\alpha}\right)^{-1}(V)$ is open in the preimage of the restriction. Then $f^{-1}(V) = \bigcup\limits_{\alpha} f^{-1}(V) \cap U_\alpha$ is open.

Finally, we turn to a nice lemma:

Lemma 1.18.9 (The Pasting Lemma). *If $X = A \cup B$ where A and B are closed, $f : A \to Y$ and $g : B \to Y$ are continuous, and $f(x) = g(x)$ for all $x \in A \cap B$, then the function $h : X \to Y$ given by*

$$h(x) = \begin{cases} f(x) & \text{if } x \in A, \\ g(x) & \text{if } x \in B \end{cases}$$

is continuous.

Proof. We show that the preimage of closed sets is closed. Let $C \subseteq Y$ be closed. Then $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$, and as f and g are continuous, $f^{-1}(C)$ is closed in A and hence closed in X, and $g^{-1}(C)$ is closed in B hence closed in X. Thus $h^{-1}(C)$ is a union of closed sets, hence closed. □

1.19 Section 19: The Product Topology

Definitions: tuple, product, box topology, product topology
Main Idea: Define the box and product topologies, and see that the box topology is a load of crap.

The following is a good way to conceptualize the concepts to come:

Think of an ordered pair $(x_1, x_2) \in X \times X$ as a function $f : \{1, 2\} \to X$ so that $f(1) = x_1$ and $f(2) = x_2$. We can generalize this to:

Definition 1.19.1. Let J be an index set. Given a set X, a J**-tuple** is a function $\overline{x} : J \to X$. Typically we write x_α for $\overline{x}(\alpha)$, $\alpha \in J$, and this is called the αth coordinate of $\overline{x}$. We also write $(x_\alpha)_{\alpha \in J}$ instead of $\overline{x}$, reminiscent of tuple notation. Finally, the set of all J-tuples is denoted X^J.

Definition 1.19.2. Given an indexed collection of sets $\{A_\alpha\}_{\alpha \in J}$, let $X = \bigcup_{\alpha \in J} A_\alpha$ and we define the **product** as

$$\prod_{\alpha \in J} A_\alpha = \left\{ (x_\alpha)_{\alpha \in J} \in X^J \mid x_\alpha \in A_\alpha \text{ for all } \alpha \in J \right\}.$$

Note here that if every A_α is equal to some set X, then $\prod_{\alpha \in J} A_\alpha$ is just X^J, the set of **all** J-tuples of elements of X.

Definition 1.19.3. We define the **box topology** on $\prod_{\alpha \in J} X_\alpha$ via the basis consisting of $\prod_{\alpha \in J} U_\alpha$ where $U_\alpha \subseteq X_\alpha$ is open. Note that this is indeed a basis, per **Definition 1.13.1**, as for every $x \in \prod_{\alpha \in J} X_\alpha$, there exists an open neighborhood U_α around every x_α whose product is a basis element in which x lies, and $\prod_\alpha U_\alpha \cap \prod_\alpha V_\alpha = \prod_\alpha U_\alpha \cap V_\alpha$, so we get even stronger than subsets.

It turns out however, that while this seems to be a straightforward way to put a topology on $\prod_{\alpha \in J} X_\alpha$, it turns out to be a very unworkable topology when $|J| = \infty$. We will see this shortly, but to that end, we define another topology which we will soon see that we prefer to use:

Definition 1.19.4. We define the **product topology**, also called the **Tychonoff topology** on $\prod_{\alpha \in J} X_\alpha$ via the basis consisting of $\prod_{\alpha \in J} U_\alpha$ where $U_\alpha \subseteq X_\alpha$ is open, and $U_\alpha = X_\alpha$ for all but finitely many α.

Note now that if J is finite, the box topology and the product topology are indeed the same, since $U_\alpha \neq X_\alpha$ for finitely many α no matter what. Now let's see these topologies in action:

Example 1.19.5. Let $X = \{0, 1\}$ with the discrete topology. Now consider $X^{\mathbf{N}}$. An element of $X^{\mathbf{N}}$ is a "∞-tuple" $(x_1, x_2, x_3, ...)$ where $x_i = 0$ or 1. We'll compare both the box and the product topology on $X^{\mathbf{N}}$.

First, we claim that the box topology is the discrete topology. We'll show that any singleton in $X^{\mathbf{N}}$ is open. Given $(x_n)_{n \in \mathbf{N}} \in X^{\mathbf{N}}$, the set $U = \prod_n \{x_n\}$ is open in the box topology, as $\{x_n\} \subseteq X$ are open, since X has the discrete topology. Then, clearly $U = \{(x_n)\}$, so this is the discrete topology.

Next, we claim that the product topology is not the discrete topology, so they are indeed different. We define $\overline{x_n} : \mathbf{N} \to X^{\mathbf{N}}$ by

$$\overline{x_n}(m) = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases}$$

So $\overline{x_1}$ is $(1, 0, 0, ...)$, $\overline{x_2}$ is $(0, 1, 0, ...)$, $\overline{x_3}$ is $(0, 0, 1, ...)$, and so on. Then we claim that $\overline{x_0}$ is a limit point of $\{\overline{x_n} \mid n \geq 1\}$, and since $\overline{x_0} \notin \overline{x_n} \mid n \geq 1\}$, $\{\overline{x_n} \mid n \geq 1\}$ is not closed. This is enough, as in the discrete topology, since all sets are open, all sets are closed.

So let's show that $\overline{x_0}$ is a limit point; recall **Definition 1.17.8** (limit points) and **Definition 1.17.13** (convergence). Every neighborhood of $\overline{x_0}$ contains a basis element $U = \prod U_n$ that contains $\overline{x_0}$, by virtue of being a basis. Then there exists N such that for $n \geq N$, $U_n = \{0, 1\} = X$, since we are in the product topology. So for $n < N$, since $\overline{x_0} \in U$, for these n it must be the case that $0 \in U_n$. And for $n \geq N$, since $U_n = X$, $\overline{x_n} \in U$, since only the nth coordinate of $\overline{x_n}$ is nonzero. This shows that $(\overline{x_n}) \to \overline{x_0}$, as desired, as $\overline{x_n}$ is only outside of U for finitely many n.

Therefore, on $X^{\mathbf{N}}$, the box topology is the discrete topology, while the product topology is not.

We can compare the two topologies in general:

Proposition 1.19.6. *The box topology is finer than the product topology. As we have already seen, if the index set is finite, then they are the same.*

We'll explore other properties of the product topology versus the box topology now.

Theorem 1.19.7. *Let $f : A \to \prod_{\alpha \in J} X_\alpha$, with $\prod_{\alpha \in J} X_\alpha$ given the product topology, be defined by $f(a) = (f_\alpha(a))_{\alpha \in J}$ for functions $f_\alpha : A \to X_\alpha$. Then f is continuous if and only if each f_α is continuous.*

Proof. We know that projection maps, $\pi_\beta : \prod_{\alpha \in J} X_\alpha \to X_\beta$ where $\pi_\beta(\overline{x}) = \overline{x}(\beta)$, the βth coordinate, are continuous. Then, since $f_\alpha = \pi_\alpha f$, if f is continuous, then so is every f_α.

Conversely, assume every f_α is continuous. We wish to show f is continuous, so we show preimages of open sets are open. For a basis element $U = \prod_{\alpha \in J} U_\alpha$, let J_U be the finite set of α such that $U_\alpha \neq X_\alpha$. Then $f^{-1}(U) = \bigcap_{\alpha \in J} {f_\alpha}^{-1}(U_\alpha) = \bigcap_{\alpha \in J_U} {f_\alpha}^{-1}(U_\alpha)$, which is a finite intersection of open sets, and hence open. Therefore, if every f_α is continuous, then f is continuous. □

Note here that this is actually false for the box topology! We see an example:

Example 1.19.8. Let $f : \mathbf{R} \to \mathbf{R}^{\mathbf{N}}$ be given by $f(t) = (t, t, t, ...)$. Then $f(t)$ is continuous in the product topology (by the **Theorem 1.19.7** above), but not in the box topology!

Consider $U = (-1, 1) \times \left(-\frac{1}{2}, \frac{1}{2}\right) \times \left(-\frac{1}{3}, \frac{1}{3}\right) \times ....$ The set U is open in the box topology, but $f^{-1}(U) = \{0\}$ which is not open!

That was a pretty reasonable function to expect to be continuous. This and many other examples demonstrate that the box topology is in general not the right tool for the job. Therefore, unless otherwise specified, from here on out, we give products the product topology.

1.20 Section 20: The Metric Topology

Definitions: metric space, epsilon ball, metric topology, metrizable, Euclidean metric, square metric, taxicab metric, uniform metric, product metric
Main Idea: This section defines several metrics.

Let's start the section with some preliminary notation.

Definition 1.20.1. We call (X, d) be a **metric space**: a set X with a metric d. (Recall that a metric is a function $d : X \times X \to [0, \infty)$ such that d is positive definite, symmetric, and satisfies the triangle inequality.

Definition 1.20.2. We call $B_d(x, \varepsilon) = \{y \in X \mid d(x, y) < \varepsilon\}$ the open ball centered at x with radius ε, or often just the **ε-ball**.

Definition 1.20.3. We define the **metric topology** via the basis $\{B_d(x, \varepsilon) \mid x \in X, \varepsilon > 0\}$. Then U is open in X if and only if for all $x \in U$, there exists $\varepsilon > 0$ such that $B_d(x, \varepsilon) \subseteq U$.

Lemma 1.20.4. *What we have described in **Definition 1.20.3** is indeed a basis.*

Proof. Recall **Definition 1.13.1**.

First, $X = \bigcup_{x \in X} B_d(x, 1)$.

Second, if $y \in B_d(x_1, \varepsilon_1) \cap B_d(x_2, \varepsilon_2)$, then take $\delta = \min\{\varepsilon - d(x_1, y), \varepsilon_2 - d(x_2, y)\}$. Then $\delta > 0$ and $\delta \leq \varepsilon_i - d(x_i, y)$ for both $i = 1$ and $i = 2$. If $z \in B_d(y, \delta)$, then $d(y, z) < \delta$ and $d(x_i, z) \leq d(x_i, y) + d(y, z) < \varepsilon_i - \delta + \delta = \varepsilon_i$. Therefore, $z \in B_d(x_1, \varepsilon_1) \cap B_d(x_2, \varepsilon_2)$; hence $B_d(y, \delta) \subseteq B_d(x_1, \varepsilon_1) \cap B_d(x_2, \varepsilon_2)$. $\square$

Definition 1.20.5. A topological space (X, τ) is **metrizable** if there exists a metrix on X inducing the topology τ.

Lemma 1.20.6. *If a space is metrizable, then it is Hausdorff.*

Proof. If $x \neq y$, then $\varepsilon = \frac{1}{2}d(x, y) > 0$. Then $B_d(x, \varepsilon) \cap B_d(y, \varepsilon) = \emptyset$. $\square$

Lemma 1.20.7. *Let d, d' be two metrics on X, inducing topologies τ and τ'. Then $\tau \subseteq \tau'$ if and only if for all $x \in X$, $\varepsilon > 0$, there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \varepsilon)$.*

As with comparing topologies, the goal is to nest one inside the other.

Proof. First, suppose $\tau \subseteq \tau'$. Then for all $x \in X$, $\varepsilon > 0$, the ε-ball $B_d(x, \varepsilon)$ is open in τ and thus open in τ'. Thus there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \varepsilon)$.

Conversely, apply **Lemma 1.13.5**. Simply put, nested bases give nested topologies, and the metric topologies are given via their bases. $\square$

Theorem 1.20.8. *Let (X, d) be a metric space and define $\overline{d} : X \times X \to \boldsymbol{R}$ by $\overline{d}(x, y) = \min\{d(x, y), 1\}$. Then $\overline{d}$ is a metric, and $\overline{d}$ induces the same topology as d.*

Things to note here: $\overline{d}$ is just the metric that caps all distances at 1. Furthermore, this theorem says that properties like boundedness, which are lost when you use the metric $\overline{d}$, and other metric space properties, do not carry any topology.

Proof. First, we show that $\overline{d}$ is a metric. Clearly it is positive definite and symmetric, since d is. To show $\overline{d}$ satisfies the triangle inequality, let $x, y, z \in X$. We wish to show that $\overline{d}(x, z) \leq \overline{d}(x, y) + \overline{d}(y, z)$. If either $d(x, y) \geq 1$ or $d(y, z) \geq 1$, then $\overline{d}(x, z) \leq 1 \leq \overline{d}(x, y) + \overline{d}(y, z)$ because one is at least 1. If, however, both $d(x, y)$ and $d(y, z)$ are less than 1, we have $\overline{d}(x, z) \leq d(x, z) \leq d(x, y) + d(y, z) = \overline{d}(x, y) + \overline{d}(y, z)$. Therefore $\overline{d}$ is a metric.

Next, we show that $\overline{d}$ induces the same topology as d. We show that each is nested in the other via **Lemma 1.20.7**. For $B_d(x, \varepsilon) \subseteq B_{\overline{d}}(x, \varepsilon)$, this is clear, since $\overline{d}(x, y) \leq d(x, y)$, so if $d(x, y) < \varepsilon$, so too $\overline{d}(x, y) < \varepsilon$. The same ε works.

For the other direction, we have $B_{\overline{d}}(x, \delta) \subseteq B_d(x, \varepsilon)$ where $\delta = \min\left\{\varepsilon, \frac{1}{2}\right\}$. To see this, if $\overline{d}(x, y) < \varepsilon$, then $d(x, y) < \frac{1}{2}$, so $\overline{d}(x, y) = d(x, y) < \varepsilon$. $\square$

Thus every metrizable space has a bounded metric, and therefore boundedness is not a topological invariant.

We now turn our attention to metrics on products. We define the three following metrics on $\mathbf{R}^n$:

Definition 1.20.9. The **Euclidean metric** on $\mathbf{R}^n$ is $d(\overline{x}, \overline{y}) = \left(\sum_{i=1}^{n} (x_i - y_i)^2\right)^{\frac{1}{2}}$. Open balls are balls.

Definition 1.20.10. The **square metric** on $\mathbf{R}^n$ is $\rho(\overline{x}, \overline{y}) = \max_{1 \leq i \leq n} |x_i - y_i|$. Open balls are squares.

Definition 1.20.11. The **taxicab metric** on $\mathbf{R}^n$ is $d_1(\overline{x}, \overline{y}) = \sum_{i=1}^{n} |x_i - y_i|$. Open balls are diamonds.

Theorem 1.20.12. *The Euclidean metric, square metric, and the taxicab metric all induce the same topology on* $\boldsymbol{R}^n$.

Proof. A proof is obvious if pictures are drawn; simply nest a ball/square/diamond inside of the others. We give an explicit proof that d and ρ induce the same topology; the rest can be similarly done.

So see that

$$\begin{aligned}\rho(\overline{x},\overline{y})^2 &= \max_{1\leq i\leq n}|x_i-y_i|^2\\ &\leq \sum_{i=1}^{n}|x_i-y_i|^2 = d(\overline{x},\overline{y})^2\\ &\leq \sum_{j=1}^{n}\max_{1\leq i\leq n}|x_i-y_i|^2 = n\rho(\overline{x},\overline{y})^2.\end{aligned}$$

Therefore $\rho(\overline{x},\overline{y}) \leq d(\overline{x},\overline{y}) \leq \sqrt{n}\rho(\overline{x},\overline{y})$, and thus $B_\rho\left(x,\frac{\varepsilon}{\sqrt{n}}\right) \subseteq B_d(x,\varepsilon) \subseteq B_\rho(x,\varepsilon)$. The other comparisons are similar. $\square$

There is another topology we can define on products:

Consider the case where we would like to define a topology on $\mathbf{R}^{\mathbf{N}}$. We could attempt to generalize d or ρ or etc. However, we run into problems; if we consider generalizing $\rho(\overline{x},\overline{y}) = \sup|x_i-y_i|$, this may not be well-defined, as the distances $|x_i-y_i|$ can increase without bound and thus the supremum would be ∞. To remedy this, we can replace the metric with its bounded metric, and thus the supremum is bounded above by 1 and defines a metric. The following definition holds in general, not just $\mathbf{R}^{\mathbf{N}}$:

Definition 1.20.13. We define the **uniform metric** on the product of metric spaces $\prod_{\alpha\in J}(X_\alpha,d_\alpha)$ by $\overline{\rho}(\overline{x},\overline{y}) = \sup\left\{\overline{d_\alpha}(x_\alpha,y_\alpha)\mid \alpha\in J\right\}$, where we recall $\overline{d_\alpha} = \min\{d_\alpha,1\}$. The uniform metric induces a topology called the uniform topology.

Theorem 1.20.14. *The product topology is coarser than the uniform topology, and the uniform topology is coarser than the box topology.*

Proof. Let $X = \prod_{\alpha\in J}X_\alpha$ be our space. Let $\overline{x}\in U = \prod_\alpha U_\alpha$ where U is a basis element in the product topology. Then there exists finitely many $\alpha_1,...\alpha_n$ for which $U_{\alpha_i}\neq X_{\alpha_i}$. Then for each i, choose $\varepsilon_i>0$ such that the $B_{\overline{d}}(x_{\alpha_i},\varepsilon_i)\subseteq U_{\alpha_i}$, as $U_{\alpha_i}\subseteq X_{\alpha_i}$ is open. Then $\varepsilon=\min\{\varepsilon_1,...\varepsilon_n\}$ and $B_{\overline{\rho}}(\overline{x},\varepsilon)\subseteq\prod_\alpha U_\alpha$ since finitely many $\varepsilon_\alpha<1$. As $B_{\overline{\rho}}(\overline{x},\varepsilon)$ is a basis element for the uniform topology, we conclude that the product topology is coarser than the uniform topology.

Next, see that $\prod_\alpha B_{\overline{d_\alpha}}\left(x_\alpha,\dfrac{\varepsilon}{2}\right)\subseteq B_{\overline{\rho}}(\overline{x},\varepsilon)$. Every $B_{\overline{d_\alpha}}\left(x_\alpha,\frac{\varepsilon}{2}\right)$ is open, so their product is open in the box topology. Therefore the uniform topology is coarser than the box topology. $\square$

Definition 1.20.15. We define the **product metric** on $\prod_{n \in \mathbf{N}} (X_n, d_n)$ by

$$D(\overline{x}, \overline{y}) = \sup \left\{ \frac{\overline{d_n}(x_n, y_n)}{n} \mid n \in \mathbf{N} \right\}.$$

Theorem 1.20.16. *The name is appropriate; the product metric D is indeed a metric and induces the product topology.*

Proof. To see that D is a metric, first note that symmetry and positive definiteness come from $\overline{d_n}$. The triangle equality follows as for all i, $\frac{\overline{d_i}(x_i,z_i)}{i} \leq \frac{\overline{d_i}(x_i,y_i)}{i} + \frac{\overline{d_i}(y_i,z_i)}{i} \leq D(\overline{x}, \overline{y}) + D(\overline{y}, \overline{z})$. Therefore $D(\overline{x}, \overline{z}) = \sup \left\{ \frac{\overline{d_i}(x_i,z_i)}{i} \right\} \leq D(\overline{x}, \overline{y}) + D(\overline{y}, \overline{z})$.

Now we show that D induces the product topology. We show each topology is nested within the other. First, let U be open in the metric topology and let $\overline{x} \in U$. Our goal is to find V in the product topology such that $\overline{x} \in V \subseteq U$. Next, choose an ε such that $B_D(\overline{x}, \varepsilon) \subseteq U$. Then choose N so that $\frac{1}{N} < \varepsilon$. Then V is the basis element in the product topology $B_{d_1}(x_1, \varepsilon) \times ... \times B_{d_N}(x_N, \varepsilon) \times X_n \times X_n \times ...$, and we claim that $V \subseteq B_D(\overline{x}, \varepsilon)$. Indeed, for $\overline{y} \in \prod_n X_n$, $\frac{\overline{d_i}(x_i,y_i)}{i} \leq \frac{1}{N}$ for $i \geq N$. Therefore, $D(\overline{x}, \overline{y}) \leq \max \left\{ \frac{\overline{d_1}(x_1,y_1)}{1}, ... \frac{\overline{d_N}(x_N,y_N)}{N}, \frac{1}{N} \right\}$, and if $\overline{y} \in V$, then $D(\overline{x}, \overline{y}) < \varepsilon$, so $V \subseteq B_D(\overline{x}, \varepsilon)$, as claimed.

Conversely, we want to show the metric topology nests inside the product topology. Let $U = \prod_n U_n$ be a basis element for the product topology, where U_i is open in X_i for $i = 1, ..., n$ and $U_i = X_i$ for all other indices. Then let $\overline{x} \in U$ and we wish to find V open in the metric topology such that $\overline{x} \in V \subseteq U$. For $i = 1, ..., n$, choose a ball $B_{d_i}(x_i, \varepsilon_i) \subseteq U_i$ with every $\varepsilon_i \leq 1$. Then we call $\varepsilon = \min \left\{ \frac{\varepsilon_i}{i} \mid i = 1, ..., n \right\}$. We claim that $V = B_D(\overline{x}, \varepsilon)$ satisfies $\overline{x} \in V \subseteq U$. To see this, let $\overline{y} \in B_D(\overline{x}, \varepsilon)$. Then for all i, $\frac{\overline{d_i}(x_i,y_i)}{i} \leq D(\overline{x}, \overline{y}) < \varepsilon$. For $i = 1, ..., n$, $\varepsilon \leq \frac{\varepsilon_i}{i}$, so $\overline{d_i}(x_i, y_i) < \varepsilon_i \leq 1$, and $\overline{y} \in U$, so $B_D(\overline{x}, \overline{y}) \subseteq U$, as desired. $\square$

1.21 Section 21: The Metric Topology (continued)

Definitions: first countable, uniform convergence
Main Idea: First countability is a restriction that lets continuity be discussed in terms of limits. Uniform convergence is nice.

Theorem 1.21.1. *Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \to Y$ is continuous if and only if for all $x \in X$, $\varepsilon > 0$, there exists $\delta > 0$ such that $f(B_{d_X}(x, \delta)) \subseteq B_{d_Y}(f(x), \varepsilon)$.*

Proof. Assume f is continuous. Then $x \in f^{-1}(B_{d_Y}(f(x), \varepsilon)) \subseteq X$, which is an open set since it is the preimage under f of an open set. Then there exists

$\delta > 0$ such that $B_{d_X}(x,\delta) \subseteq f^{-1}(B_{d_Y}(f(x),\varepsilon))$, and hence $f(B_{d_X}(x,\delta)) \subseteq B_{d_Y}(f(x),\varepsilon)$.

Conversely, assume for all $x \in X$, $\varepsilon > 0$, there exists $\delta > 0$ such that $f(B_{d_X}(x,\delta)) \subseteq B_{d_Y}(f(x),\varepsilon)$. We wish to show f is continuous. Let $V \subseteq Y$ be open and $x \in f^{-1}(V) \subseteq X$. See that $f(x) \in V$, so there exists $\varepsilon > 0$ such that $B_{d_Y}(f(x),\varepsilon) \subseteq V$. Then by assumption, there exists $\delta > 0$ such that $f(B_{d_X}(x,\delta)) \subseteq B_{d_Y}(f(x),\varepsilon) \subseteq V$. Thus $x \in B_{d_X}(x,\delta) \subseteq f^{-1}(V)$, so $f^{-1}(V)$ is open. Thus, f is continuous. $\square$

We are now motivated by the following question: outisde of topology, we define continuity as if $(x_n) \to x$, then $(f(x_n)) \to f(x)$. When does this definition hold in topological spaces? We introduce the following definition which suffices:

Definition 1.21.2. A topological space X is **first countable**, or satisfies the first countability axiom, if for all $x \in X$, there exists a countable neighborhood basis $\{U_n\}_{n\in\mathbf{N}}$ of x such that if V is any neighborhood of x, then there exists $n \in \mathbf{N}$ such that $U_n \subseteq V$.

Example 1.21.3. All metric spaces (X,d) are first countable. Fix a point $x \in X$; then the countable neighborhood basis is the set of $U_n = B_d\left(x, \frac{1}{n}\right)$.

Example 1.21.4. Let's see an example of a space that is **not** first countable. Let $X = \{0,1\}^{\mathbf{R}}$. Recall from **Definition 1.19.1** that this is the set of all $\mathbf{R}$-tuples; i.e., $\{0,1\}^{\mathbf{R}} = \{f \mid f : \mathbf{R} \to \{0,1\}\}$. Give $\{0,1\}$ the discrete topology, and X of course the product topology. We claim that X is not first countable. We show a point that has no countable neighborhood basis.

Consider $\overline{x} \in X$ the zero function; i.e., $\overline{x}(\alpha) = 0$ for all $\alpha \in \mathbf{R}$. Then suppose there exists a countable neighborhood basis $\{U_n\}_{n\in\mathbf{N}}$ of $\overline{x}$. For each $n \in \mathbf{N}$, let $J_n \subseteq \mathbf{R}$ be the finite set where $U_n = \prod U_{n,\alpha}$ where $U_{n,\alpha}$ are open in $\{0,1\}$ and $U_{n,\alpha} = \{0,1\}$ if $\alpha \notin J_n$.

Then for $\alpha_0 \in \mathbf{R} \setminus \bigcup\limits_{n\in\mathbf{N}} J_n$, the set $V_0 = \prod V_\alpha$, where V_α is $\{0\}$ if $\alpha = \alpha_0$ and $\{0,1\}$ otherwise, is a neighborhood of $\overline{x}$. However, $U_n \nsubseteq V_0$ for any n, because $U_{n,\alpha_0} = \{0,1\}$.

Note that $\{0,1\}^{\mathbf{R}}$ is not metrizable under the product topology. We'll soon see that that's a necessary condition.

For now, let's return to the motivation of first countability; our goal was to use it to describe continuity in terms of limits. We have that in the following lemma:

Lemma 1.21.5. *If X is first countable and $x \in \overline{A}$, then there exists a sequence $(x_n) \subseteq A$ such that $(x_n) \to x$.*

Proof. As X is first countable, let $\{U_n\}_{n\in\mathbf{N}}$ be a countable neighborhood basis of x. Then set $B_n = U_1 \cap U_2 \cap ... \cap U_n$. As this is a finite intersection of open sets, B_n is open still, and furthermore, $B_{n+1} \subseteq B_n$. Then for all $n \in \mathbf{N}$, fix $x_n \in A \cap B_n$ We claim that this creates the sequence (x_n) that converges to x.

To see this, given U any neighborhood of x, we have $U_N \subseteq U$ for some N by first countability. If $n \geq N$, $x_n \in B_n \subseteq B_N \subseteq U_N \subseteq U$. Thus the tail of (x_n) stays inside the neighborhood U of x, and $(x_n) \to x$, as claimed. $\square$

Corollary 1.21.6. *The converse is always true, even if X is not first countable: If there exists a sequence $(x_n) \subseteq A$ such that $(x_n) \to x$, then $x \in \overline{A}$.*

Proof. x is a limit point of A, and $\overline{A}$ contains the limit points of A. (See **Definitions 1.17.8, 1.17.13**, and **Theorem 1.17.11**.) $\square$

Theorem 1.21.7. *Let $f : X \to Y$. If f is continuous and $(x_n) \to x$, then $(f(x_n)) \to f(x)$.*

Furthermore, let X be first countable. If $(x_n) \to x$ implies that $(f(x_n)) \to f(x)$, then f is continuous.

Proof. First, suppose f is continuous and $(x_n) \to x$. We wish to show that $(f(x_n)) \to f(x)$. Let $V \subseteq Y$ be a neighborhood of $f(x)$. Then $U = f^{-1}(V)$ is a neighborhood of x. Then the tail of the sequence remains in U: there exists $N \in \mathbf{N}$ such that if $n \geq N$, $x_n \in U$. Hence $f(x) \in V$ for $n \geq N$. Therefore, $(f(x_n)) \to f(x)$.

Now, suppose X is first countable and that for all $(x_n) \to x$, $(f(x_n)) \to f(x)$. We'll show that $f(\overline{A}) \subseteq \overline{f(A)}$. (Recall **Theorem 1.18.3**.)

Let $x \in \overline{A}$ and thus $f(x) \in f(\overline{A})$. We need to show $f(x) \in \overline{f(A)}$. By **Lemma 1.21.5**, there exists $(x_n) \subseteq A$ such that $(x_n) \to x$. Then by hypothesis, $(f(x_n)) \to f(x)$. Since every $x_n \in A$, every $f(x_n) \in f(A)$, so we have $f(x) \in \overline{f(A)}$ by **Corollary 1.21.6**, as desired. Therefore, f is continuous. $\square$

Definition 1.21.8. Let $f_n : X \to Y$ be a sequence of continuous functions from X into a metric space Y. We say that (f_n) **converges uniformly** to $f : X \to Y$ if for all $\varepsilon > 0$, there exists $N \in \mathbf{N}$ such that if $x \in X$ and $n \geq N$, then $d(f_n(x), f(x)) < \varepsilon$.

Theorem 1.21.9. *If $f_n : X \to Y$ is continuous for all $n \in \mathbf{N}$ and $(f_n) \to f$ uniformly, then $f : X \to Y$ is continuous.*

Proof. Let $V \subseteq Y$ be open and let $x_0 \in f^{-1}(V) \subseteq X$. We need to show $f^{-1}(V)$ is open; we procure a neighborhood U of x_0 such that $f(U) \subseteq V$.

There exists $\varepsilon > 0$ such that $B_d(f(x_0), \varepsilon) \subseteq V$. As $(f_n) \to f$ uniformly, there exists $N \in \mathbf{N}$ such that if $n \geq N$ then for all $x \in X$, $d(f_n(x), f(x)) < \frac{\varepsilon}{3}$. As f_N is continuous, there is a neighborhood U of x_0 such that $f_N(U) \subseteq B_d\left(f_N(x_0), \frac{\varepsilon}{3}\right) \subseteq Y$.

We claim that this U works; that $f(U) \subseteq B_d(f(x_0), \varepsilon) \subseteq V$. To see this, let $x \in U$, and see that

$$\begin{aligned} d(f(x), f(x_0)) &\leq d(f(x), f_N(x)) + d(f_N(x), f_N(x_0)) + d(f_N(x_0), f(x_0)) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &= \varepsilon, \end{aligned}$$

so $f(x) \in B_d(f(x_0), \varepsilon)$, as desired. $\square$

1.22 Section 22: The Quotient Topology

Definitions: quotient map, quotient topology, quotient space
Main Idea: The biggest thing to take away from this section is that you've spent a year developing an intuition for quotients, so don't let these unintuitive definitions bog you down. Quotient spaces glue bits together, quotient groups collapse relations down to the identity, and quotient maps are terrifying and unhelpful, but related to these.

Definition 1.22.1. A surjective function[1] $p : X \to Y$ is called a **quotient map** if for all $U \subseteq Y$, $p^{-1}(U)$ is open in X if and only if U is open in Y.

Some remarks that may or may not be helpful: quotient maps are continuous, as **Definition 1.18.2** says that $U \subseteq Y$ open implies $p^{-1}(U) \subseteq X$ is open. However, quotient maps are **not** open maps (see **Definition 1.18.5**), since an open map sends $U \subseteq X$ open to $f(U) \subseteq Y$ open, but a quotient map only requires open sets in X that are the preimage of a set in Y to have open image. Let's see some examples of quotient maps:

Example 1.22.2. Let $p : [0, 1] \to S^1 = \{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 = 1\}$ be $p(t) = (\cos(2\pi t), \sin(2\pi t))$. The map p is not an open map; see **Example 1.18.8**. Quickly, $[0, x) \subseteq [0, 1]$ is open, but $p([0, x)) \subseteq S^1$ is not open.

However, p is a quotient map. Any open set $V \subseteq S^1$ gives an open preimage $p^{-1}(V) \subseteq [0, 1]$.

Definition 1.22.3. Let X be a topological space, A be a set, and $p : X \to A$ a surjection. There is a unique topology on A such that p is a quotient map. We call such a topology on A the **quotient topology**.

In other words, a set $U \subseteq A$ is open if and only if $p^{-1}(U) \subseteq X$ is open.

Lemma 1.22.4. *The quotient topology on A is a topology.*

Proof. We show it satisfies all the requirements in **Definition 1.12.1**.

First, $\emptyset$ and A are open because $p^{-1}(\emptyset) = \emptyset \subseteq X$ open and $p^{-1}(A) = X \subseteq X$ open.

Next, the topology is closed under unions: $p^{-1}\left(\bigcup_{\alpha \in J} U_\alpha\right) = \bigcup_{\alpha \in J} p^{-1}(U_\alpha)$.

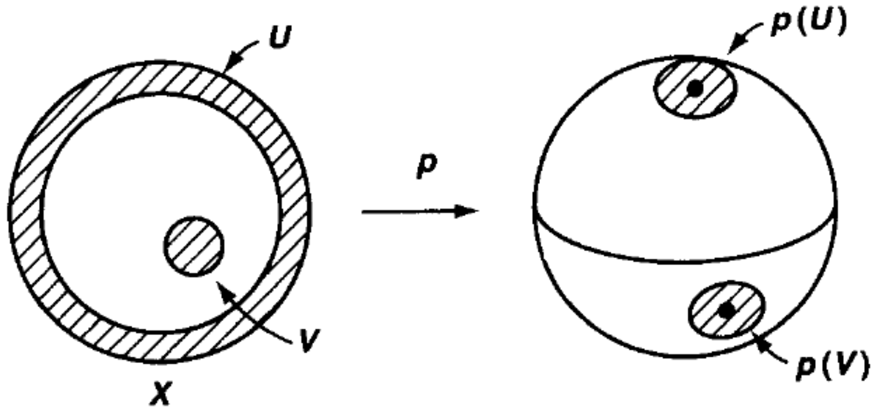
And finite intersections: $p^{-1}\left(\bigcap_{i=1}^{n} U_i\right) = \bigcap_{i=1}^{n} p^{-1}(U_i)$. □

There is a special, frequently used instance of the quotient topology:

Definition 1.22.5. Let X be a topological space and let $\sim$ be an equivalence relation on X. Then there is a canonical surjection $p : X \to X/\sim$ via $p(x) = [x]$, and p induces the quotient topology on $X/\sim$. The space $X/\sim$ is called the **quotient space**, or often **identification** or **decomposition space**, of X, since we identify points equivalent under the relation $\sim$.

[1] the image $f(p)$ is all of the target space Y, even if two points map to the same point

Example 1.22.6. Let $X = \{\overline{x} \in \mathbf{R}^2 \mid \|\overline{x}\| \leq 1\} = D^2$, the unit disk. Then define $\sim$ on X by $\overline{x} \sim \overline{y}$ if either $\overline{x} = \overline{y}$ or $\|\overline{x}\| = \|\overline{y}\| = 1$. In other words, every point is its own equivalence class, except the boundary all identifies/collapses. This means $p : D^2 \to S^2$, and the boundary of D^2 maps to say, the north pole, as in the following picture.



Note that every quotient map arises in this way; we can define $\sim$ on X by $x_1 \sim x_2$ if $p(x_1) = p(x_2)$. This is good, because quotient spaces are the most intuitive way to think about quotients.

Theorem 1.22.7. *Let $p : X \to Y$ be a quotient map. Let $g : X \to Z$ be such that $g(x_1) = g(x_2)$ if $p(x_1) = p(x_2)$. Then there exists an induced map $f : Y \to Z$ which is continuous if and only if g is continuous.*

Proof. We give an explicit construction of f. Let $f : Y \to Z$ be $f(y) = g(x)$ if $y = p(x)$. As p is a quotient map, f doesn't depend on choice of x, since $x \in p^{-1}(\{y\})$. Then $g = fp$ by construction.

$$\begin{tikzcd} X \arrow[d, "p"'] \arrow[dr, "g"] & \\ Y \arrow[r, dotted, "f"] & Z \end{tikzcd}$$

Now, assume f is continuous. Since $g = fp$ and f and p are continuous, g is continuous.

Now, assume g is continuous. Then let $V \subseteq Z$ be open. We see that $f^{-1}(V) \subseteq Y$ is open, as p is a quotient map, and $p^{-1}(f^{-1}(V)) = (fp)^{-1}(V) = g^{-1}(V)$ is open. $\square$

1.23 Section 23: Connected Spaces

Definitions: separation, connected, totally disconnected
Main Idea: Connectedness is a pretty intuitive property. Since we define connectedness as "there does not exist a separation," to prove a space is connected,

many proofs assume a separation exists and reach a contradiction.

Definition 1.23.1. A **separation** of a topological space X is a pair of nonempty sets $C, D \subseteq X$ such that $C \cap D = \emptyset$ and $C \cup D = X$.

Definition 1.23.2. We say X is **connected** if there does not exist a separation of X.

Notice that if C and D form a separation, then C and D are clopen (closed and open), as $C = X \setminus D$ and $D = X \setminus C$. It's therefore equivalent to say that a space X is connected if and only if the only clopen subsets are X and $\emptyset$.

Example 1.23.3. Let $Y = [-1, 0) \cup (0, 1] \subseteq \mathbf{R}$. Then Y is not connected in the subspace topology, as $[-1, 0) = (-2, 0) \cap Y$ is open, and also $[-1, 0) = [-1, 0] \cap Y$ is closed.

Example 1.23.4. $\mathbf{Q} \subseteq \mathbf{R}$ is not connected. See that $\mathbf{Q} = [(-\infty, \sqrt{2}) \cap \mathbf{Q}] \cup [(\sqrt{2}, \infty) \cap \mathbf{Q}]$. Here is a separation.

Example 1.23.5. Let $X = \{0, 1\}^{\mathbf{N}}$ with the discrete topology on $\{0, 1\}$. Then X is not connected. Let $C = \{0\} \times \prod_{n \geq 1} \{0, 1\}$. C is open in the product topology. Similarly, $D = \{1\} \times \prod_{n \geq 1} \{0, 1\}$ is open in the product topology. See that C is the tuples that start with 0 and D the tuples that start with 1, so $X = C \cup D$.

Lemma 1.23.6. *If C and D form a separation of X and $Y \subseteq X$ is a connected subspace, then either $Y \subseteq C$ or $Y \subseteq D$.*

You should already have a picture in your head because this is so obvious. Let's give the proof.

Proof. Suppose not, and Y is not a subset of C or D. Then $Y \cap C$ and $Y \cap D$ are disjoint, nonempty, and open in Y, and $Y = (Y \cap C) \cup (Y \cap D)$. This separates Y, but Y is connected. This is a contradiction. $\square$

Definition 1.23.7. We say that X is **totally disconnected** if the only connected subspaces of X are singletons.

Example 1.23.8. We show that $X = \{0, 1\}^{\mathbf{N}}$, discrete topology on $\{0, 1, \}$, is totally disconnected.

Given $\overline{x}, \overline{y} \in X$ where $\overline{x} \neq \overline{y}$, we construct a separation C and D where $\overline{x} \in C$ and $\overline{y} \in D$. By the arbitrary nature of $\overline{x}$ and $\overline{y}$, this will show any two points can be separated, and hence only singletons can be connected.

Since $\overline{x} \neq \overline{y}$, there exists a coordinate N such that $x_N \neq y_N$. Analogous to **Example 1.23.5**, let $C = \prod_{n<N} \{0, 1\} \times \{x_N\} \times \prod_{n>N} \{0, 1\}$ and let $D = \prod_{n<N} \{0, 1\} \times \{y_N\} \times \prod_{n>N} \{0, 1\}$. Then $\overline{x} \in C$, $\overline{y} \in D$, and $X = C \cup D$.

Theorem 1.23.9. *Suppose Y is a connected subspace of X, and $\{Y_\alpha\}_{\alpha\in J}$ is a collection of connected subspaces of X such that $Y \cap Y_\alpha \neq \emptyset$ for all α. Then $X_0 = Y \cup \left(\bigcup_{\alpha\in J} Y_\alpha\right)$ is connected.*

Once again, the picture is handy for intuition. Below we see it and then give the proof.

Proof. Suppose to the contrary that $C, D \subseteq X_0$ is a separation. Then for all $\alpha \in J$, $Y_\alpha \subseteq C$ or $Y_\alpha \subseteq D$, and similarly, $Y \subseteq C$ or $Y \subseteq D$, by **Lemma 1.23.6**. Without loss of generality, we have $Y \subseteq C$. Since $Y \cap Y_\alpha \neq \emptyset$ for all α, every $Y_\alpha \subseteq C$ as well. Then $X_0 \subseteq C$, so C, D could not have been a separation of X_0. $\square$

Theorem 1.23.10. *Suppose $A \subseteq X$ is connected and $A \subseteq B \subseteq \overline{A}$. Then B is connected. In other words, adding some or all limit points does not affect connectivity.*

Proof. Suppose instead that $B = C \cup D$ where C, D are a separation of B. Then without loss of generality, $A \subseteq C$, as A is connected. Then $D = B \cap U$ for some $U \subseteq X$ open. For $b \in D$, U is a neighborhood of b in X that is disjoint from C, and hence disjoint from A. Therefore $b \notin \overline{A}$, but this contradicts the fact that C, D separated B. $\square$

Theorem 1.23.11 (The Intermediate Value Theorem). *If X is connected and $f : X \to Y$ is continuous, then $f(X) \subseteq Y$ is continuous.*

Proof. Suppose $f(X) \subseteq Y$ is not connected. Then there exist $U, V \subseteq Y$ open such that $C = f(X) \cap U$ and $D = f(X) \cap V$ are a separation of $f(X)$. But then, as f is continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are a separation of X, a contradiction. $\square$

Theorem 1.23.12. *If X and Y are connected, then so is $X \times Y$.*

Proof. Let X and Y be connected spaces. Let $x_0 \in X$; then the function $f : Y \to \{x_0\} \times Y$ via $f(y) = (x_0, y)$ is continuous. Since Y is connected and f continuous, the **Intermediate Value Theorem 1.23.11** gives that $\{x_0\} \times Y$ is connected. Similarly, fix a $y \in Y$ and $g_y : X \to X \times \{y\}$ via $g_y(x) = (x, y)$ shows that $X \times \{y\}$ is connected for all y.

Now, $X \times Y = \{x_0\} \times Y \cup \bigcup_{y \in Y} (X \times \{y\})$. See that for all y, $\{x_0\} \times Y \cap X \times \{y\}$ is nonempty because (x_0, y) lies in it. Therefore, by **Theorem 1.23.9**, $X \times Y$ is connected, as desired. $\square$

This theorem gives us that finite products of connected spaces are connected; just induct on the factors. It remains to be seen if infinite products of connected spaces are connected: in the product topology, yes. But for the box topology, no, of course not, the box topology is worthless.

Example 1.23.13. In the box topology, $\mathbf{R}^{\mathbf{N}}$ is not connected. We claim that if C is the set of bounded sequences and D the set of unbounded sequences, then C and D separate $\mathbf{R}^{\mathbf{N}}$. Certainly they are disjoint and their union is the entire space, so we just need to show that C and D are open in the box topology.

Let $\overline{x} \in \mathbf{R}^{\mathbf{N}}$. Then set $U = (x_1 - 1, x_1 + 1) \times (x_2 - 1, x_2 + 1) \times ...$, open (in the box topology). If $\overline{x}$ is bounded, then $U \subseteq C$, and if not, $U \subseteq D$.

1.24 Section 24: Connected Subspaces of the Real Line

Definitions: path, path connected
Main Idea: Path connectedness is straightforward. Path connected implies connected, but the converse is not true; take the Topologist's Sine Curve.

Before we begin, note a few key (defining, often) properties of $\mathbf{R}$:

- The least upper bound property: $\sup A$ exists if A is bounded above.
- Archimedean property: if $x, y \in \mathbf{R}$ such that $x < y$, then there exists $z \in \mathbf{R}$ such that $x < z < y$.

If you're seeking to generalize the results of these sections, any space with these properties should be sufficient (along with anything else we use in the hypotheses of a theorem or during a proof).

Theorem 1.24.1. ***R*** *is connected.*

Proof. In the same nature as the contradiction proofs, let $U, V \subseteq \mathbf{R}$ be nonempty disjoint open subsets. Here, we show that $U \cup V$ cannot be all of $\mathbf{R}$.

Fix $u \in U$ and $v \in V$, and take $u < v$. Then let $U_0 = U \cap [u, v]$, and let $V_0 = V \cap [u, v]$. Now, let $c = \sup U_0$. Thus $u < c \leq v$. We'll show that $c \notin U_0$ and $c \notin V_0$; thus $c \notin U \cup V$ and hence $U \cup V$ isn't all of $\mathbf{R}$.

Suppose $c \in V_0$. As V_0 is open in $[u, v]$ by the subspace topology, there exists $d \in [u, v]$ such that $(d, c] \subseteq V_0$. But then d is a smaller upper bound to U_0, which contradicts the fact that c is. So $c \notin V_0$.

Then suppose $c \in U_0$. As U_0 is open in $[u, v]$ by the subspace topology and $v \notin U_0$, there exists $d \in [u, v]$ such that $[c, d) \subseteq U_0$. Now there exists z such that $c < z < d$, so c is not an upper bound to U_0. Therefore, $c \notin U_0$, and therefore $c \notin U \cup V$, as desired. $\square$

Corollary 1.24.2. *The following subsets of* $\boldsymbol{R}$ *are connected:*

1. *open intervals* (a, b),
2. *open rays* $(-\infty, a)$ *or* (a, ∞),
3. *closed rays* $(-\infty, a]$ *or* $[a, \infty)$,
4. *half open intervals* $[a, b)$ *or* $(a, b]$, *and*
5. *closed intervals* $[a, b]$.

Proof. 1. and 2. are both homeomorphic to $\mathbf{R}$, and connectedness is a topological invariant.

For 4. and 5., we know that (a, b) is connected, and by **Theorem 1.23.10**, so are $[a, b)$, $(a, b]$, and $[a, b]$, since $\overline{(a, b)} = [a, b]$.

For 3., since $\overline{(-\infty, a)} = (-\infty, a]$ and $\overline{(a, \infty)} = [a, \infty)$, we can again use **Theorem 1.23.10**. □

Corollary 1.24.3. *Since* $\boldsymbol{R}$ *is connected and the finite product of connected spaces is connected,* $\boldsymbol{R}^n$ *is connected.*

Corollary 1.24.4. $\boldsymbol{R}^{\boldsymbol{N}}$ *with the product topology is connected.*

(It is not in the box by **Example 1.23.13**.)

Proof. Let $\widetilde{\mathbf{R}^N}$ be tuples (x_n) where $x_n = 0$ for $n > N$. Then $\widetilde{\mathbf{R}^N} \subseteq \mathbf{R}^{\mathbf{N}}$.

Let $\mathbf{R}^\infty = \bigcup_{N \in \mathbf{N}} \widetilde{\mathbf{R}^N}$, which can be realized as tuples which are eventually 0.

$\mathbf{R}^\infty$ is connected, as each $\widetilde{\mathbf{R}^N}$ meets $\overline{0}$. In the product topology, $\overline{\mathbf{R}^\infty} = \mathbf{R}^{\mathbf{N}}$, so $\mathbf{R}^{\mathbf{N}}$ is connected. □

Definition 1.24.5. Given $x, y \in X$, a **path** from x to y is a continuous function $p : [a, b] \to X$ such that $p(a) = x$ and $p(b) = y$. We often just take $p : [0, 1] \to X$.

Definition 1.24.6. A space X is **path connected** if for all $x, y \in X$, there exists a path from x to y.

Example 1.24.7. The space $\mathbf{R}^n$ is path connected. Given $x, y \in \mathbf{R}^n$, one explicit path is $p : [0, 1] \to \mathbf{R}^n$, $p(t) = (1 - t)x + ty$.

Theorem 1.24.8. *If* X *is path connected, then it is connected.*

Proof. We show that X not connected implies that X is not path connected. Let C, D be a separation of X and $p : [a, b] \to X$ a continuous path. Since $[a, b]$ is connected, so is $p([a, b])$. Therefore $p([a, b]) \subseteq C$ or $p([a, b]) \subseteq D$. So there does not exist a path from any $x \in C$ to any $y \in D$. Thus X is not path connected, as desired. □

Example 1.24.9. This is one of the quintessential examples of a space connected but not path connected: the Topologist's Sine Curve.

Let $S = \left\{\left(x, \sin\left(\frac{1}{x}\right)\right) \in \mathbf{R}^2 \mid 0 < x \leq 1\right\} = f((0,1])$ where $f : (0,1] \to \mathbf{R}$, $f(x) = \sin\left(\frac{1}{x}\right)$. The map f is continuous and thus S is connected (as $(0,1]$ is), and S is path connected (trace the image of f), *at this stage*.

However, since S is connected, $\overline{S}$ is also connected, but we see that $\overline{S}$ is not path connected. Once again, the picture should be convincing enough, but we make an explicit proof:

We first describe the closure of S: $\overline{S} \setminus S = \{(0,y) \mid -1 \leq y \leq 1\}$ which we call Y. Now, suppose $p : [0,1] \to \overline{S}$ where $p(0) = (0,0)$. We claim that $p^{-1}(Y) \subseteq [0,1]$ is nonempty and clopen, and thus since $[0,1]$ is connected, $p([0,1]) \subseteq Y$. In the language we will define soon, $\overline{S}$ has two path components, and here we are stuck on the component Y.

To show the claim, first, nonempty is clear, since $p^{-1}(0,0) = 0 \in [0,1]$.

Next, as $Y \subseteq \overline{S} \subseteq \mathbf{R}$ is closed, $p^{-1}(Y)$ is closed, since p_* is continuous and thus the preimage of closed sets is closed.

Finally, we show $p^{-1}(Y)$ is open. Take $t \in p^{-1}(Y)$ such that $p(t) = (0,y)$. Then let $U = \overline{S} \cap ((-\delta,\delta) \times (y-\delta, y+\delta))$ for $0 < \delta < \frac{1}{2}$. Therefore, U is an open square smaller than the amplitude of the Topologist's Sine Curve intersected with the Topologist's Sine Curve.

Then $V = p^{-1}(U)$ is open and contains t. There exists an open connected subset I_t such that $t \in I_t \subseteq V$ (in the language to come, this is a path component of U). We want to show that $p(I_t) \subseteq Y$.

For all $n \in \mathbf{N}$ we can find a separation C_n, D_n of U where $C_{n+1} \subseteq C_n$ and $\bigcap\limits_{n\in\mathbf{N}} C_n \subseteq Y$. As $p(I_t)$ is connected and meets C_n for all n, $p(I_t) \subseteq C_n$ for all n, so $p(I_t) \subseteq \bigcap\limits_{n\in\mathbf{N}} C_n \subseteq Y$, as desired.

1.25 Section 25: Components and Local Connectedness

Definitions: component, path component, locally connected, locally path connected

Main Idea: Components/path components can be pictured fairly well. You have a good intuition for the picture of local connectedness even if the definition is still a bit unwieldy. Note that locally path connected implies locally connected, and we have several theorems that compare connected/path connected to locally connected/locally path connected.

Definition 1.25.1. Given a space X, define an equivalence relation $\sim$ by $x \sim y$ if there exists $C \subseteq X$ a connected set such that $x, y \in C$. The equivalence classes are called **(connected) components** of X.

Lemma 1.25.2. *This is in fact an equivalence relation.*

Proof. Reflexive is easy; x is in the same component as itself x. A single point is connected.

Symmetric is clear; x and y are in the same component if and only if y and x are in the same component. The same C works.

For transitivity, let $x, y \in A$ and $y, z \in B$ with A, B connected subspaces of X. We need to show that there is a connected set containing both x and z. Indeed, $x, z \in A \cup B$, which is connected by **Theorem 1.23.9**, since A and B each meet y, and $\{y\}$ is connected. $\square$

We now list some properties of components with proof:

Lemma 1.25.3.
1. *Each component is connected.*
2. *Components $C_1 \cap C_2 \neq \emptyset$ if and only if $C_1 = C_2$.*
3. *If $A \subseteq X$ is connected, then there exists a component $C \subseteq X$ such that $A \subseteq C$.*
4. *Components are closed.*

Proof.
1. Fix $x \in C$. Then for all $y \in C$, there exists $A_y \subseteq X$ a connected subspace of X such that $x, y \in A_y$. Then $C = \bigcup_{y \in C} A_y$ is connected as each A_y meets x.
2. Components are equivalence classes. If there is any overlap at all, they must be the same class.
3. For all $x, y \in A$, $x \sim y$, so they belong to the same equivalence class. In other words, components are maximal connected subspaces, which is the best way of thinking of them.
4. If C is a component, then C is connected, and $\overline{C}$ is connected. By 3., $\overline{C} \subseteq C$, but since $\overline{C} = C \cup C'$, $C = \overline{C}$. $\square$

Note that though components are indeed closed, they need not be open. Consider $\mathbf{Q} \subseteq \mathbf{R}$. Then the components of $\mathbf{Q}$ are singletons, and not open in the subspace topology.

However, if there are only finitely many components $C_1, ...C_n$, then components are closed, since $X \setminus C_k = \bigcup_{i \neq k} C_i$. This is a finite union of closed sets, hence closed, so $X \setminus C_k$ is closed, and thus C_k is open for all $k = 1, ..., n$.

Definition 1.25.4. Given a space X, define an equivalence relation $\sim$ by $x \sim y$ if there exists a path from x to y. The equivalence classes are called **path components** of X.

Lemma 1.25.5. *This is indeed an equivalence relation.*

Proof. Reflexive is clear; the constant path $p : [a, b] \to X$, $p(t) = x$ works.

For symmetry, if $p : [0, 1] \to X$ is a path from x to y, then define $q : [0, 1] \to X$ from y to x by $q(t) = p(1 - t)$. This is just reversing the path p.

Finally, transitivity is just composing the paths; if $p : [0, 1] \to X$ is a path from x to y and $q : [1, 2] \to X$ is a path from y to z, then define $r : [0, 2] \to X$ by

$$r(t) = \begin{cases} p(t) & \text{if } 0 \leq t \leq 1, \\ q(t) & \text{if } 1 \leq t \leq 2. \end{cases}$$

By the **Pasting Lemma 1.18.9**, r is continuous and hence a path from x to z. □

It can be shown that properties 1., 2., and 3. of components in **Lemma 1.25.3** hold for path components as well. However, path components need not be closed, nor, for that matter open. To that end, we consider:

Example 1.25.6. Recall the Topologist's Sine Curve $\overline{S}$ from **Example 1.24.9**; recall in particular that $Y = \overline{S} \setminus S$. As we alluded to in that example, the path components are exactly Y and S. S is open, since it is the image of an open set, and not closed, since its limit points are Y, so therefore $Y = \overline{S} \setminus S$ must be closed and not open.

A quick remark is that path components are contained in components, since path connectedness is a stronger requirement than connectedness.

Definition 1.25.7. A space X is **locally connected** if for all $x \in X$ and neighborhoods U of x, there exists a neighborhood V of x, which is connected, such that $V \subseteq U$.

Definition 1.25.8. A space X is **locally path connected** if for all $x \in X$ and neighborhoods U of x, there exists a neighborhood V of x, which is path connected, such that $V \subseteq U$.

We remark that locally path connected implies locally connected, by the same proof as the global case.

A natural question is: how does local connectedness/path connectedness compare to (global) connectedness/path connectedness?

Example 1.25.9. First a simple example: any open subset of $\mathbf{R}^n$ is locally connected and locally path connected.

Example 1.25.10. The Topologist's Sine Curve $\overline{S}$ (**Example 1.24.9**) is connected as we have seen, but it is not locally connected.

Consider $\overline{S} \cap (-\delta, \delta) \times (y - \delta, y + \delta)$, the Topologist's Sine Curve intersected with an open square. Then for some x in this intersection, any neighborhood looks like disjoint vertically oriented bars, not connected.

Example 1.25.11. Now consider a joined-up Topologist's Sine Curve, where we create a path from Y to S that simply joins the ends up. Now this space is path connected, because we've created a path between the two path components, but just like **Example 1.25.10**, this space is not locally connected, or for that matter, locally path connected.

So how do we compare connectedness and local connectedness? Path connectedness and local path connectedness?

Theorem 1.25.12. *X is locally (path) connected if and only if for all $U \subseteq X$ open, each (path) component of U is open in X.*

We note that the theorem follows the same proof whether we are dealing with paths or not. Thus we give the proof that X is locally connected exactly when every component of U is open, and the proof with paths is the same.

Proof. First, assume that X is locally connected. Then let $U \subseteq X$ be open and $C \subseteq U$ a component of U. We wish to show that $C \subseteq X$ is open. Let $x \in C$. Then there exists a connected neighborhood $V \subseteq U$ of x, since X is locally connected. Therefore $V \subseteq C$, so C is open.

Conversely, fix $x \in X$ and let $U \subseteq X$ be a neighborhood of x, and $x \in C \subseteq U$ open. We wish to show that X is locally connected. Well, C is exactly a connected neighborhood of x in U, and thus X is locally connected. □

Theorem 1.25.13. *If X is locally path connected, then the set of components and the set of path components are the same.*

Proof. We know by a quick remark that path components are contained in components. In other words, let $P \subseteq C$ where P is a path component and C a connected component. We need to show $C = P$.

As X is locally path connected, P must be open in C by **Theorem 1.25.12**. Now we show P is closed, and since C is connected, $P = C$ (since the only clopen sets are C and $\emptyset$.

To see that P is closed, let Q be the union of the other path components in C. Then Q is a union of open sets, hence open, and since $C \setminus P = Q$, P is closed. Therefore, $P = C$, as desired. □

1.26 Section 26: Compact Spaces

Definitions: cover, compact, finite intersection property
Main Idea: Compactness is alright. Throwing on other properties to your space gains you more results.

Definition 1.26.1. A collection $\mathcal{A}$ of open subsets is called an open **cover** if $X = \bigcup_{U \in \mathcal{A}} U$.

Definition 1.26.2. A space X is **compact** if for every open cover $\mathcal{A}$, there exist finitely many $U_1, ..., U_n \in \mathcal{A}$ such that $X = \bigcup_{i=1}^{n} U_i$. The collection is called a finite subcover, and hence X is compact if every cover has a finite subcover.

Example 1.26.3. $\mathbf{R}$ is not compact. The cover $\mathcal{A} = \{(n-1, n+1) \mid n \in \mathbf{Z}\}$ has no finite subcover, as each $U_n = (n-1, n+1) \in \mathcal{A}$ meets any integer only once.

Example 1.26.4. $[0,1]$ is compact. We don't yet have the machinery to prove this, but it is by Heine-Borel, to come (**Theorem 1.27.2**).

Lemma 1.26.5. *$Y \subseteq X$ is compact if and only if every convering of Y by open sets in X has a finite subcovering.*

Proof. First, assume Y is compact. Then let $\mathcal{A} = \{U_\alpha \mid U_\alpha \subseteq X \text{ is open}\}$ be a cover of Y; i.e., $Y \subseteq \bigcup_{U_\alpha \in \mathcal{A}} U_\alpha$. We want to show Y is covered by finitely many U_α. Indeed, $\mathcal{A}_Y = \{U_\alpha \cap Y \mid U_\alpha \in \mathcal{A}\}$ is an open cover of Y in Y, so there is a finite subcover $U_1 \cap Y, ..., U_n \cap Y$ such that $Y = \bigcup_{i=1}^{n} U_i \cap Y$. Therefore $Y \subseteq \bigcup_{i=1}^{n} U_i$ is a finite subcover of U_is open in X.

Conversely, assume $\mathcal{A} = \{U_\alpha\}$ is an open cover of Y with $U_\alpha \subseteq Y$ open. We wish to show that there is a finite subcover of $\mathcal{A}$. Consider $\mathcal{A}' = \{V_\alpha\}$ where $U_\alpha = V_\alpha \cap Y$, so $V_\alpha \subseteq X$ is open. Then by hypothesis there are finitely many $V_1, ...V_n$ that cover Y. Then $U_i = V_i \cap Y$, where $i = 1, ..., n$, is a finite subcover of $\mathcal{A}$, as desired. □

Theorem 1.26.6. *Closed subspaces of compact spaces are compact.*

Proof. Let X be compact and let $Y \subseteq X$ be closed. We need to show that Y is compact. Let $\mathcal{A}$ be a cover of Y by open sets in X. Since Y is closed, $X \setminus Y$ is open, and $\mathcal{A} \cup \{X \setminus Y\}$ is an open cover of X. As X is compact, there exist finitely many $U_1, ..., U_n \in \mathcal{A} \cup \{X \setminus Y\}$ such that $Y \subseteq X = \bigcup_{i=1}^{n} U_i$. There's your finite subcover of Y. □

What about the converse? Is every compact set closed? Unfortunately not:

Example 1.26.7. In the finite complement topology (recall **Definition 1.12.6**; U is open when U is empty or the complement of U is finite) on $\mathbf{R}$, every subset is compact! To see this, let $A \subseteq \mathbf{R}$ and let $\mathcal{A}$ be an open cover of A. Choose an open U_{α_1} in $\mathcal{A}$. Then U_{α_1} misses only finitely many points $x_1, ...x_n$, and thus we can choose at most n more, but critically, only finitely many more, U_{α_i} to cover the rest of $\mathbf{R}$.

So not every compact set is closed. However:

Theorem 1.26.8. *Compact subspaces of Hausdorff spaces are closed.*

Proof. Let X be Hausdorff and $Y \subseteq X$ be compact. We wish to show that Y is closed. To that end, we show that $X \setminus Y$ is open.

Fix $x_0 \in X \setminus Y$. For each $y \in Y$, there are disjoint neighborhoods U_y and V_y of x_0 and y, respectively, as X is Hausdorff. Note further that $x_0 \in \bigcap_{y \in Y} U_y \subseteq X \setminus Y$. We claim that $\bigcap_{y \in Y} U_y$ is open, and thus $X \setminus Y$ is open.

To see this, since Y is compact, there exist finitely many $y_1, ..., y_n$ such that $Y \subseteq \bigcup_{i=1}^{n} V_{y_i}$. Then $x_0 \in \bigcap_{i=1}^{n} U_{y_i} \subseteq X \setminus Y$. This is a finite intersection of open sets and hence open, so it just remains to be see that indeed $\bigcap_{i=1}^{n} U_{y_i} \subseteq X \setminus Y$.

For this, let $z \in Y$. Then $z \in V_{y_i}$ for some i, so $z \notin U_{y_i}$. Thus $z \notin \bigcap_{i=1}^{n} U_{y_i}$, and we are done. □

Lemma 1.26.9. *If X is Hausdorff and $Y \subseteq X$ is compact, then for all $x_0 \notin Y$, there exist disjoint open sets U and V such that $x_0 \in U$ and $Y \subseteq V$. In other words, in a Hausdorff space, we can separate closed/compact spaces from a point.*

Proof. Let $U = \bigcap_{i=1}^{n} U_{y_i}$ and $V = \bigcup_{i=1}^{n} V_{y_i}$. The machinery of **Theorem 1.26.8** is enough to show this works. □

This lemma actually shows that if X is compact and Hausdorff, then X is regular (see **Definition 1.31.1**, to come).

Theorem 1.26.10. *If X is compact and $f : X \to Y$ is continuous, then $f(X) \subseteq Y$ is compact.*

Proof. As X is compact, given a cover $\mathcal{A}$, there exists a finite subcover. Then $f(\mathcal{A})$ covers $f(X)$, and the finite subcover of $\mathcal{A}$ under the image of f is a finite subcover of $f(X)$. □

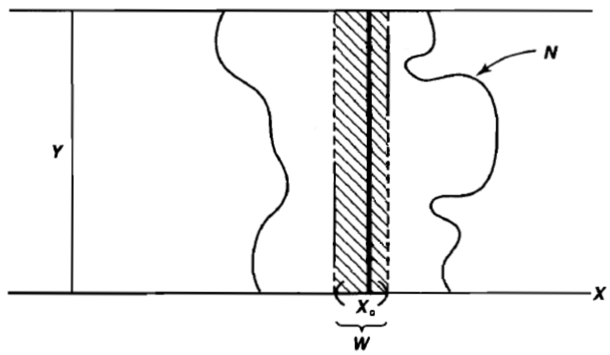
Theorem 1.26.11. *Let $f : X \to Y$ be a continuous bijection. If X is compact and Y is Hausdorff, then f is a homeomorphism.*

Proof. Since f is a continuous bijection, we can show that f is an open/closed map, which will do the trick. We show that f is a closed map; f sends closed sets to closed sets.

First, as f is a bijection, $f(X) = Y$, so by **Theorem 1.26.10**, Y is compact. So now suppose $A \subseteq X$ is closed. Then since X is compact, by **Theorem 1.26.6**, A is compact as well. By **Theorem 1.26.10**, $f(A) \subseteq Y$ is compact. By **Theorem 1.26.8**, since Y is Hausdorff and $f(A)$ is compact, $f(A)$ is closed. Thus f is a closed map, and f is a homeomorphism, as desired. □

Lemma 1.26.12 (The Tube Lemma). *Suppose Y is compact. Suppose $N \subseteq X \times Y$ is open. Suppose $\{x_0\} \times Y \subseteq N$ for some $x_0 \in X$. Then there exists $W \subseteq X$ a neighborhood of x_0 such that $W \times Y \subseteq N$.*

The picture is rather nice here. It's literally a tube.



Proof. For all $y \in Y$ there exists $U_y \subseteq X$, open neighborhood of x_0, and $V_y \subseteq Y$, open neighborhood of y, such that $(x_0, y) \subseteq U_y \times V_y \subseteq N$. Then $\{V_y\}$ is a cover of Y, so as Y is compact, there exist $y_1, ..., y_n$ such that $Y = \bigcup_{i=1}^{n} V_{y_i}$. Set $W = \bigcap_{i=1}^{n} U_{y_i}$; we claim this works. See that if $(x, y) \in W \times Y$, there exists y_i such that $y \in V_{y_i}$. This implies that $(x, y) \in W \times V_{y_i} \subseteq U_{y_i} \times V_{y_i} \subseteq N$. □

Theorem 1.26.13. *If X and Y are compact, so is $X \times Y$.*

Clearly, we can then induct to get that the product of finitely many compact spaces is compact. It is also true for the infinite case; see **Theorem 1.37.3** to come.

Proof. Let $\mathcal{A}$ be an open cover of $X \times Y$; we procure a finite subcover.

Given $x_0 \in X$, there exists $U_1, ..., U_n \in \mathcal{A}$ such that $\{x_0\} \times Y \subseteq \bigcup_{i=1}^{n} U_i = N$. Then N is an open set containing $\{x_0\} \times Y$.

We can therefore apply the **Tube Lemma 1.26.12** to get a neighborhood $W_{x_0} \subseteq X$ of x_0 such that $W_{x_0} \subseteq N$. Then $\{W_{x_i}\}$ is an open cover of X, so there exist $x_1, ..., x_m$ such that $X = \bigcup_{i=1}^{m} W_{x_i}$. Thus $\bigcup_{i=1}^{m} W_{x_i} \times Y = X \times Y$, and each $W_{x_i} \times Y$ is convered by finitely many sets in $\mathcal{A}$, and as there are finitely many of them, we have that $X \times Y$ is covered by finitely many sets, as desired. □

Definition 1.26.14. A collection $\mathcal{C}$ of sets in X has the **finite intersection property** if for all finite subcollections $\mathcal{C}' \subseteq \mathcal{C}$, $\bigcap_{C \in \mathcal{C}'} C \neq \emptyset$.

Theorem 1.26.15. *A space X is compact if and only if for every collection $\mathcal{C}$ of closed sets with the finite intersection property, $\bigcap_{C \in \mathcal{C}} C \neq \emptyset$.*

Proof. We know that X is not compact if and only if there exists an open cover $\mathcal{A} = \{U_\alpha\}_{\alpha \in J}$ such that $U_{\alpha_1} \cup ... \cup U_{\alpha_n} \neq X$ for all finite subcollections. This is equivalent to saying that there exists a collection of closed sets $\mathcal{C} = \{C_\alpha\}_{\alpha \in J}$ where $C_\alpha = X \setminus U_\alpha$ such that $C_{\alpha_1} \cap ... \cap C_{\alpha_n} \neq \emptyset$ for all finite subcollections, yet $\bigcap_{\alpha \in J} C_\alpha = \emptyset$. □

1.27 Section 27: Compact Subspaces of the Real Line

Definitions: set distance
Main Idea: There are a few more theorems and topics in Munkres that aren't in our notes. As it is now, this is a very basic real analysis section in the language of topology, like the other section that discusses $\mathbf{R}$.

Theorem 1.27.1. *A closed interval in* $\boldsymbol{R}$ *is compact.*

Proof. Any closed interval $[a, b]$ is homeomorphic to $[0, 1] \subseteq \mathbf{R}$, so we prove it here, using the finite intersection property and **Theorem 1.26.15**.

Let $\mathcal{C}$ be a collection of closed subsets of $[0, 1]$ with the finite intersection property. For a finite subcollection $\mathcal{A} \subseteq \mathcal{C}$, let $b_{\mathcal{A}} = \inf \bigcap_{C \in \mathcal{A}} C \in [0, 1]$. Note that $b_{\mathcal{A}} \in \bigcap_{C \in \mathcal{A}} C$, since every neighborhood of $b_{\mathcal{A}}$ must contain a point in the intersection, which is closed.

Now let $b = \sup\{b_{\mathcal{A}} \mid \mathcal{A} \subseteq \mathcal{C} \text{ is a finite subcollection}\}$. We claim that $b \in \bigcap_{C \in \mathcal{C}} C$.

To see this, fix $C_0 \in \mathcal{C}$, and we will show that $b \in C_0$. Let $\mathcal{A}_1, \mathcal{A}_2, ..., \mathcal{A}_n, ...$ be finite subcollections such that $b \geq b_{\mathcal{A}_n} \geq b - \frac{1}{n}$. Then define ${\mathcal{A}_n}' = \mathcal{A}_n \cup \{C_0\}$, so it is still the case that $b \geq b_{{\mathcal{A}_n}'} \geq b - \frac{1}{n}$. Therefore, every neighborhood of b contains $b_{{\mathcal{A}_n}'}$ for large enough n. As $b_{{\mathcal{A}_n}'} \in \bigcap_{C \in {\mathcal{A}_n}'} C \subseteq C_0$, we have that $b \in \overline{C_0} = C_0$, since C_0 is closed. As C_0 was arbitrary, $b \in \bigcap_{C \in \mathcal{C}} C$. Thus $\bigcap_{C \in \mathcal{C}} C \neq \emptyset$, so by **Theorem 1.26.15**, we are done. □

Theorem 1.27.2 (Heine-Borel). *A subset* $A \subseteq \boldsymbol{R}^n$ *is compact if and only if* A *is closed and bounded.*

Proof. Let A be compact. Then since $\mathbf{R}^n$ is Hausdorff, compact sets are closed, so A is closed. To see that A is bounded, we show that any compact subspace A of a metric space X is bounded:

Fix $x \in X$. Take the collection $\{B_d(x, i)\}_{i=1}^{\infty}$. Then $\bigcup_{i=1}^{\infty} B_d(x, i)$ is an open cover of X. Thus $A \subseteq \bigcup_{i=1}^{\infty} B_d(x,i)$, and since A is compact, there is a finite subcover $A \subseteq \bigcup_{i=1}^{n} B_d(x,i)$. But these sets are nested; i.e., $\bigcup_{i=1}^{n} B_d(x,i) = B_d(x,n)$. Therefore, A is bounded in the ball of radius n. Then apply this proof to $X = \mathbf{R}^n$ to get that A is bounded.

Conversely, let A be closed and bounded; we wish to show A is compact. Since A is bounded, $A \subseteq [-M,M]^n$ for some $M > 0$. As $[-M,M]^n$ is compact and A is closed, A is compact. □

Definition 1.27.3. Let (X,d) be a metric space, $x \in X$ and $A \subseteq X$ nonempty. We define the **distance from** x **to** A by $d : X \times A \to \mathbf{R}$, where $d(x,A) = \inf\{d(x,a) \mid a \in A\}$.

Lemma 1.27.4. *$d(-,A) : X \to \boldsymbol{R}$ is continuous.*

Proof. Fix A and let $x,y \in X$. Then for all $a \in A$, $d(x,A) \leq d(x,a) \leq d(x,y) + d(y,a)$. Therefore, $d(x,A) - d(x,y) \leq \inf d(y,a) = d(y,A)$, and thus $|d(x,A) - d(y,A)| \leq d(x,y)$. Thus if $x \to y$, then $d(x,A) \to d(y,A)$. □

Lemma 1.27.5 (**The Lebesgue Covering Lemma**). *Let (X,d) be a compact metric space, and let $\mathcal{A}$ be an open cover of X. Then there exists $\delta > 0$ such that if $Y \subseteq X$ and $\operatorname{diam}(Y) < \delta$, then there exists $U \in \mathcal{A}$ such that $Y \subseteq U$.*

We call δ the Lebesgue number of $\mathcal{A}$. Think of umbrellas at a football game as the cover $\mathcal{A}$. If you are Y and your shoulders are narrower than δ, then you fit entirely under an umbrella.

Proof. Let $\mathcal{A}$ be an open cover of X and assume that $X \notin \mathcal{A}$.[2] Then, as X is compact, there exist $U_1, ..., U_n \in \mathcal{A}$ such that $X \subseteq \bigcup_{i=1}^{n} U_i$.

Let $C_i = X \setminus U_i$. Define the continuous function $f : X \to \mathbf{R}$ by $f(x) = \frac{1}{n}\sum_{i=1}^{n} d(x,C_i)$. Then, fix $x \in X$, and so $x \in U_i$ for some $i = 1, ..., n$. Thus there exists $\varepsilon > 0$ such that $B_d(x,\varepsilon) \subseteq U_i$. Therefore $d(x,C_i) \geq \varepsilon$, so $f(x) \geq \frac{\varepsilon}{n}$.

Now let $\delta = \inf f(x)$. Since X is compact and f is continuous, by the **Extreme Value Theorem**, δ is attained and $\delta > 0$. We claim that this δ works.

Suppose $Y \subseteq X$ and $\operatorname{diam}(Y) < \delta$. Then fix an $x_0 \in Y$, so $Y \subseteq B_d(x_0,\delta)$. Then $\delta \leq f(x_0) \leq \max\{d(x_0,C_i) \mid i = 1, ..., n\} = d(x_0,C_m)$ for some m. Therefore, $Y \subseteq B_d(x_0,\delta) \subseteq X \setminus C_m = U_m$, as desired. □

[2]If $X \in \mathcal{A}$, then any $\delta > 0$ works, because you can take $U = X$. So we assume not and show that the proof still holds.

1.28 Section 28: Limit Point Compactness

Definitions: limit point compact, sequentially compact
Main Idea: Understand the definitions of limit point compactness and sequentially compactness. Touch on a few examples and theorems.

Definition 1.28.1. A space X is **limit point compact** if every infinite subset has a limit point.

Theorem 1.28.2. *If X is compact, then X is limit point compact.*

Proof. Let X be compact. Let $A \subseteq X$, and let A', the set of limit points of A, be empty. We'll show that A must be finite.

Since $\overline{A} = A \cup A' = A$, A is closed. Then, since $A' = \emptyset$, for each $a \in A$, there exists a neighborhood U_a of a such that $U_a \cap A = \{a\}$. Then $\{U_a\}_{a \in A} \cup X \setminus A$ is an open cover of X. Since X is compact, this cover must have a finite subcover. However, each U_a only contains a single element $a \in A$. Therefore, A must be finite. □

What about the converse? If X is limit point compact, must X be compact? We see now that the answer is no:

Example 1.28.3. Let $X = \mathbf{N} \times \{a, b\}$. Give $\mathbf{N}$ the discrete topology, and give $\{a, b\}$ the trivial topology (recall **Definition 1.12.4**; the only open sets are $\emptyset$ and $\{a, b\}$).

We can see that X is not compact: the collection $\mathcal{A} = \{U_n = \{n\} \times \{a, b\} \mid n \in \mathbf{N}\}$ covers X, but there is no finite subcover, as each n only meets one U_n.

However, we show that X is limit point compact. If $(n, a) \in A \subseteq X$, then $(n, b) \in A'$, because every neighborhood of (n, b) contains (n, a). And symmetrically, If $(n, b) \in B \subseteq X$, then $(n, a) \in B'$. Therefore, **every** nonempty subset of X has a limit point, so in particular, every infinite subset has a limit point. Thus, X is limit point compact, and therefore limit point compact **does not imply** compact!

Definition 1.28.4. A space X is **sequentially compact** if every sequence $(x_n) \subseteq X$ has a convergent subsequence (x_{n_k}).

Theorem 1.28.5. *Let X be a metric space. The following are equivalent:*

1. *X is compact.*
2. *X is limit point compact.*
3. *X is sequentially compact.*

Proof. We've shown that 1. implies 2. in any space, not just metric spaces.

To show 2. implies 3., assume X is limit point compact. Then let $(x_n) \subseteq X$ be a sequence, and we wish to show that there exists a convergent subsequence (x_{n_k}). Let $A = \{x_n \mid n \in \mathbf{N}\}$. If A is finite, then for some $x_i \in (x_n)$, x_i is repeated infinitely often, so there is a constant covergent subsequence (x_i).

If A is not finite, since X is limit point compact, there exists a limit point $x \in A'$. Then let $A_1 = \{n \in \mathbf{N} \mid x_n \in B_d(x, 1)\}$. We see that A_1 is infinite, because if not, $B_d(x,1) \setminus A_1$ is a neighborhood of x disjoint from A, contradicting the fact that $x \in A'$. For this same reason, $A_k = \left\{n \in \mathbf{N} \mid x_n \in B_d\left(x, \frac{1}{k}\right)\right\}$ is also infinite. Now we build the convergent subsequence. Let $n_1 = \min A_1$, and inductively define $n_k = \min A_k \setminus \{n_1, ..., n_{k-1}\}$. Then $(x_{n_k}) \to x$, since for any tail, the sequence remains in the $\frac{1}{k}$ ball.

Finally, we show that 3. implies 1. Assume that X is sequentially compact; we wish to show X is compact. We first claim that any open cover $\mathcal{A}$ has a Lebesgue number $\delta > 0$.

> To prove this claim, suppose not. Then for all $n > 0$, there exists $C_n \subseteq X$ with $\operatorname{diam}(C_n) < \frac{1}{n}$ such that $C_n \not\subseteq U$ for any $U \in \mathcal{A}$. Choose for every n an $x_n \in C_n$; our goal is to show that $C_* \subseteq U$, a contradiction. As $x_n \in C_n$ and X is sequentially compact, the sequence (x_n) constructed has a convergent subsequence (x_{n_k}) that converges to some a. Since $\mathcal{A}$ is a cover, $a \in U$ for some $U \in \mathcal{A}$. Then, as U is open, there exists $\varepsilon > 0$ such that $B_d(a, \varepsilon) \subseteq U$. Now let k be large enough so that $\frac{1}{n_k} < \frac{\varepsilon}{2}$, so $x_{n_k} \in B_d\left(a, \frac{\varepsilon}{2}\right)$. Therefore $C_{n_k} \subseteq B_d\left(a, \frac{\varepsilon}{2}\right) \subseteq U$, the contradiction we sought.

We next claim that for all $\varepsilon > 0$, there exist $x_1, ...x_n$ such that $X = \bigcup\limits_{i=1}^{n} B_d(x_i, \varepsilon)$.

> To prove this claim, suppose not. We construct a sequence without a convergent subsequence, contradicting the fact that X is sequentially compact. Fix $x_1 \in X$. Inductively choose $x_{n+1} \in X \setminus \bigcup\limits_{i=1}^{n} B_d(x_i, \varepsilon)$, possible as this difference is nonempty by assumption. Then the sequence (x_n) does not have a convergent subsequence. This is because $d(x_n, x_m) \geq \varepsilon$ for all $n \neq m \in \mathbf{N}$.

With the claims proven, we now know that X has a Lebesgue number $\delta_{\mathcal{A}} > 0$ and that $X = \bigcup\limits_{i=1}^{n} B_d(x_i, \varepsilon)$. Let $\mathcal{A}$ be a cover of X, δ its Lebesgue number. Let $\varepsilon = \frac{\delta}{2}$ and $x_1, ...x_n \in X$ such that $X = \bigcup\limits_{i=1}^{n} B_d(x_i, \varepsilon)$. Recall that our goal is to show that X is compact, i.e., that there exists a finite subcover of $\mathcal{A}$.

We have $\operatorname{diam}(B_d(x_i, \varepsilon)) < \delta$. Thus, for each x_i, there exists $U_i \in \mathcal{A}$ such that $B_d(x_i, \varepsilon) \subseteq U_i$. As there are finitely many x_i, $\{U_i\}_{i=1}^{n}$ is a finite subcover, and we are done. $\square$

So in metric spaces, all these notions of compactness are equivalent. However, in general we can't compare compactness and sequentially compactness. Any pathological examples need not to be metric spaces.

1.29 Section 29: Local Compactness

Definitions: locally compact, compactification
Main Idea: Compactification is a nice property; make sure you're aware of the hypotheses (locally compact and Hausdorff).

Theorem 1.29.1. *Suppose X is Hausdorff and let $x \in X$. Then the following are equivalent:*

1. *There exists $C \subseteq X$ compact and $V \subseteq X$ a neighborhood of x such that $V \subseteq C$.*
2. *There exists $V \subseteq X$ a neighborhood of x such that $\overline{V}$ is compact.*
3. *For all neighborhoods $U \subseteq X$ of x, there exists $W \subseteq U$ a neighborhood of x such that $\overline{W}$ is compact.*

Proof. Assume 1. and we show 2. Since X is Hausdorff and C is compact, C is closed, and since $V \subseteq C$, $\overline{V} \subseteq C$. Closed subsets of compact spaces are compact (**Theorem 1.26.6**), so $\overline{V}$ is compact.

For 2. implies 3, let V be a neighborhood of x such that $\overline{V}$ is compact. Let $U \subseteq X$ be an arbitrary neighborhood of x. We construct a $W \subseteq U$ neighborhood of x such that $\overline{W}$ is compact. Let $W = U \cap V$. Then $\overline{W} = \overline{U \cap V} \subseteq \overline{U} \cap \overline{V} \subseteq \overline{V}$, so since $\overline{V}$ is compact and $\overline{W}$ is a closed subset of $\overline{V}$, $\overline{W}$ is compact.

Finally, assume 3. to show 1. For an arbitrary neighborhood of x U, let $W \subseteq U$ be a neighborhood of x where $\overline{W}$ is compact. Then let $C = \overline{W}$, and $V = W$. Then C is compact and $V \subseteq C$ is a neighborhood of x. ∎

Condition 1. of **Theorem 1.29.1** is important enough to get its own definition:

Definition 1.29.2. Let X be a space. We call X **locally compact** if for all $x \in X$, there exists $C \subseteq X$ compact and $V \subseteq X$ a neighborhood of x such that $V \subseteq C$.

Note immediately that compact implies locally compact.

Example 1.29.3. See that $\mathbf{R}^n$ is locally compact. Given $\overline{x} = (x_1, ..., x_n)$, then let $C = [x_1 - \delta, x_1 + \delta] \times ... \times [x_n - \delta, x_n + \delta]$. Then choose the neighborhood $V = \left(x_1 - \frac{\delta}{2}, x_1 + \frac{\delta}{2}\right) \times ... \times \left(x_n - \frac{\delta}{2}, x_n + \frac{\delta}{2}\right)$.

Example 1.29.4. Let our space be $\mathbf{R}^2$ with the Paris metric. We define the Paris metric[3] by

$$d_{\text{Paris}}((r_1, \theta_1), (r_2, \theta_2)) = \begin{cases} |r_1 - r_2| & \text{if } \theta_1 = \theta_2, \\ r_1 + r_2 & \text{if } \theta_1 \neq \theta_2. \end{cases}$$

[3] "In France, every train goes through Paris, $(0, 0)$."

Now, consider $B_{d_{\text{Paris}}}((0,0),\varepsilon)$. We see that $\left\{\left(\frac{\varepsilon}{2},\theta\right)\right\}_{\theta\in[0,2\pi)} \subseteq B_{d_{\text{Paris}}}((0,0),\varepsilon)$. The set $\left\{\left(\frac{\varepsilon}{2},\theta\right)\right\}_{\theta\in[0,2\pi)}$ doesn't have a limit point, as the Paris distance between any two points is ε. Thus it is an infinite set without a limit point, and therefore no compact set can contain it. So no neighborhood of $(0,0)$ is contained in a compact set, and thus $\mathbf{R}^2$ with the Paris metric is not locally compact.

Theorem 1.29.5. *X is locally compact and Hausdorff if and only if there exists a compact Hausdorff space Y where X is a subspace of Y and $Y \setminus X$ is a single point.*

In other words, if we start with X locally compact and Hausdorff, we can add on exactly one point to get $X \cup \{*\}$ compact. We'll define this in a moment (**Definition 1.29.6**).

Proof. Let Y be compact and Hausdorff, let X be a subspace of Y, and let $Y \setminus X$ be a singleton. We wish to show that X is locally compact and Hausdorff. Hausdorff is simple; since $X \subseteq Y$, Y Hausdorff implies that X is. For local compactness, let $x \in X$ and let $y \in Y \setminus X$. Then as X is Hausdorff, there exist disjoint neighborhoods of x and y, U and V. Then $C = Y \setminus V$ is compact, since Y is compact Hausdorff, and $U \subseteq C$. So X is locally compact.

Conversely, let X be locally compact and Hausdorff. Let $Y = X \cup \{\infty\}$. The picture to have in mind is $\mathbf{R} \cup \{\infty\} = S^1$. Let's describe the topology on Y. Open sets in Y are of the following two types:

- Type I: open sets are $U \subseteq X$ open, or
- Type II: open sets are $Y \setminus C$ where $C \subseteq X$ is compact.

Let's confirm that this is a topology (**Definition 1.12.1**):

- $\emptyset$ is open, since $\emptyset \subseteq X$ is open (Type I).
- Y is open, since $Y = Y \setminus \emptyset$ and $\emptyset$ is compact (Type II).
- The union of open sets is open, which we show by considering cases:
  - If we take a union of Type I sets, then $\bigcup U_{\mathrm{I}} = U \subseteq X$ and by the topology on X, U is open in X, so U is a Type I open set in Y.
  - If we take a union of Type II sets, then $\bigcup U_{\mathrm{II}} = \bigcup (Y \setminus C_\alpha) = Y \setminus \bigcap C_\alpha = Y \setminus C$, which is Type II in Y.
  - If we take a union of Type Is and Type IIs, then $\bigcup U_{\mathrm{I}} \cup \bigcup U_{\mathrm{II}} = U \cup (Y \setminus C) = Y \setminus (C \setminus U)$. This is Type II open in Y, since $U \subseteq C$ is closed and C is compact, so $C \setminus U$ is compact.
- The finite intersection of open sets is open, again via cases. We show a single intersection, and induction will show a finite amount:
  - If we take $U_{\mathrm{I}} \bigcap U_{\mathrm{I}}$, this is open in X and hence Type I in Y.
  - If we take $U_{\mathrm{II}} \bigcap U_{\mathrm{II}} = Y \setminus C_1 \cap Y \setminus C_2 = Y \setminus (C_1 \cup C_2)$, Type II.

- If we take $U_{\mathrm{I}} \bigcap U_{\mathrm{II}} = U \cap (Y \setminus C) = U \cap (X \setminus C)$, which is Type I since X is Hausdorff. C is compact, hence closed, so $X \setminus C$ is open.

Let's return to the proof. We've just defined $Y = X \cup \{\infty\}$ and confirmed that Y has the given topology. We're trying to show that Y is compact Hausdorff and that X is a subspace of Y, since $Y \setminus X = \{\infty\}$ is clear.

We first show that X is a subspace, by showing the subspace topology and the Type I/Type II topology are included in each other. First, let $U \subseteq X$ be open. Then $U \subseteq Y$ is open (Type I). Obviously, $U = U \cap X$. Thus the subspace topology on X is finer than the Type I/Type II topology we've described for Y. Next, let $U \subseteq Y$ be open and Type I. Then $U \cap X = U$ is open in X. Otherwise let $Y \setminus C \subseteq Y$ be open and Type II. Then $Y \setminus C \cap X = X \setminus C$ is open, since C is compact and X is Hausdorff (so C is closed). Thus the Type I/Type II topology is finer than the subspace topology. Therefore, the Type I/Type II topology is the subspace topology, so X is a subspace of Y.

Finally, we show that Y is compact and Hausdorff. To see Hausdorff, let $x_1, x_2 \in X$. Since X is Hausdorff, there exist disjoint neighborhoods U_1 and U_2 of x_1 and x_2 in X. Therefore U_1 and U_2 are disjoint neighbhoords of x_1 and x_2 in Y. Otherwise, if $x \in X$, we need to show that x can be "housed off" from ∞ in Y. Since X is locally compact, there exists $C \subseteq X$ compact and a neighborhood U of x such that $U \subseteq C$. Then U and $Y \setminus C$ are disjoint neighborhoods of x and ∞.

And to see Y compact, let $\mathcal{A}$ be an open cover of Y; we procure a finite subcover. Since ∞ must be covered, $Y \setminus C \in \mathcal{A}$ for some $C \subseteq X$ compact. As C is compact, there exist $U_1, ..., U_n \in \mathcal{A}$ such that $C \subseteq \displaystyle\bigcup_{i=1}^{n} U_i$.

Hence $Y = (Y \setminus C) \cup \displaystyle\bigcup_{i=1}^{n} U_i$. Hooray! ☐

Definition 1.29.6. If Y is a compact Hausdorff space and X is a proper subspace of Y such that $\overline{X} = Y$, then Y is said to be a **compactification** of X. Furthermore, if $Y \setminus X$ is a singleton as above in **Theorem 1.29.5**, then we say Y is the **one-point compactification** of X.

Corollary 1.29.7. *The one-point compactification of X is unique.*

Proof. Suppose Y' is compact and Hausdorff, $X \subseteq Y'$, and $Y' \setminus X = \{\infty\}$. Then define $h : Y' \to Y$ by

$$h(x) = \begin{cases} x & \text{if } x \in X, \\ \infty' & \text{if } x = \infty. \end{cases}$$

Then h is a bijection, and because Y' is compact and Y is Hausdorff, we just need continuity to show that h is a homeomorphism (recall that this is **Theorem 1.26.11**). To see this, we need to show that both Type I and Type II sets open in Y are open in Y' under h^{-1}.

For Type I, let $U \subseteq X$ be open. Then $h^{-1}(U) = U$ is open in X, and since $X \subseteq Y'$, $U \subseteq Y'$ is open.

For Type II, let $Y \setminus C$ be open where C is compact. Then $h^{-1}(Y \setminus C) = Y' \setminus C$. Since $C \subseteq Y'$ is compact and Y' is Hausdorff, C is closed, so $Y' \setminus C$ is open in Y'.

Therefore, h is continuous, hence a homeomorphism, and Y is homeomorphic to Y'. $\square$

Example 1.29.8. If $X = \mathbf{R}^2$, then $Y = S^2$. Take the plane and "pull up its corners to the point at infinity" à la the Riemann sphere.

Example 1.29.9. If X is already compact (and Hausdorff still), then $Y = X \cup \{\infty\}$ where $\{\infty\}$ is open.

Example 1.29.10. Let's see what happens if X is compact but not Hausdorff. Consider $X = (\mathbf{R}^2, d_{\text{Paris}})$, compact but not Hausdorff. Compact subsets of X are finite unions of closed ray segments emanating from the origin.

Then every neighborhood of ∞ contains points arbtrarily close to the origin. There's uncountably many rays there, so not compact.

Corollary 1.29.11. *Suppose X is Hausdorff. Then X is locally compact if and only if for all $x \in X$ and neighborhoods $U \subseteq X$ of x, there exists a neighborhood $W \subseteq X$ of x such that $\overline{W}$ is compact and $\overline{W} \subseteq U$.*

Proof. Note that "if for all $x \in X$ and neighborhoods $U \subseteq X$ of x, there exists a neighborhood $W \subseteq X$ of x such that $\overline{W}$ is compact and $\overline{W} \subseteq U$, then X is Hausdorff" is just strengthening **Theorem 1.29.1**.

Therefore, we just need to show one direction. Assume X is Hausdorff. Then consider $Y = X \cup \{\infty\}$, the one-point compactification of X. Let $x \in X$ and let $U \subseteq X$ be a neighborhood of x. We need to show there exists $W \subseteq X$, a neighborhood of x, such that $\overline{W}$ is compact and $\overline{W} \subseteq U$.

We know $C = Y \setminus U$ is closed in Y, hence compact. Thus there exist disjoint open sets V and W in Y such that $x \in W$ and $C \subseteq V$. Then $\overline{W} \subseteq Y \setminus V$, since $W \subseteq Y \setminus V$ and $Y \setminus V$ is closed. Furthermore, $Y \setminus V \subseteq Y \setminus C = U$. Therefore $\overline{W} \subseteq U$ as desired. $\square$

Corollary 1.29.12. *Suppose X is locally compact and Hausdorff. Suppose $A \subseteq X$ is open or closed. Then A is locally compact.*

Proof. If A is closed, let $x \in A$. Let $C \subseteq X$ be compact and let $V \subseteq X$ be a neighborhood of x such that $V \subseteq C$. Consider $A \cap C$ and $V \cap A$. $V \cap A$ is a neighborhood of x in A, and $A \cap C$ is a closed subset of X compact, so $A \cap C$ is compact. Therefore A is locally compact.

If A is open, let $x \in A$. By **Corollary 1.29.11**, there exists a neighborhood $W \subseteq A$ of x such that $\overline{W}$ is compact and $\overline{W} \subseteq A$. Then if we take $C = \overline{W}$, A is locally compact. $\square$

1.30 Section 30: The Countability Axioms

Definitions: first countable, second countable
Main Idea: First countable is a countable neighborhood basis. Second countable is a countable basis. Second countable, in particular in conjunction with other properties, leads to nicely behaved spaces.

Definition 1.30.1. We say that X is **first countable** if for all $x \in X$, there exists a countable neighborhood basis $\{U_n\}_{n\in\mathbf{N}}$. In other words, U_n is a neighborhood of x for all $n \in \mathbf{N}$, and if V is a neighborhood of x, then there exists $n \in \mathbf{N}$ such that $U_n \subseteq V$.

Example 1.30.2. All metric spaces are first countable. Take $U_n = B_d\left(x, \frac{1}{n}\right)$ for all n.

Proposition 1.30.3. *If X is first countable, then*

1. $(x_n) \subseteq A$ *converges to x if and only if* $x \in \overline{A}$.

   (Note that "If $(x_n) \to x$, then $x \in \overline{A}$" is always true. First countability gives the other direction.)

2. *A function $f : X \to Y$ is continuous if and only if for all convergent sequences $(x_n) \subseteq X$ converging to x, $(f(x_n)) \to f(x)$.*

   (Once again "If f is continuous, then $(x_n) \to x$ implies $(f(x_n)) \to f(x)$" is always true. First countability gives the other direction.)

Definition 1.30.4. A space X is **second countable** if there exists a countable basis $\{B_n\}_{n\in\mathbf{N}}$.

Note that if X is second countable, then X is first countable.

Example 1.30.5. $\mathbf{R}^n$ is second countable; take $\{B_d(\overline{p}, r) \mid \overline{p} \in \mathbf{Q}^n, r \in \overline{Q}_+\}$.

Example 1.30.6. $(\mathbf{R}^2, d_{\text{Paris}})$ is first countable, because it is a metric space, but it is not second countable. To see this, observe:

Theorem 1.30.7. *A metric space X is second countable if and only if there exists $A \subseteq X$ such that $\overline{A} = X$. This is called "X is separable."*

Proof. If X is separable, $\{B_d(a, r) \mid a \in A, r \in \mathbf{Q}_+\}$ is your countable basis. If there exists a countable basis, for each B_n, choose an $x_n \in B_n$. Let $D = \{x_n\}$. Then we show $\overline{D} = X$. Certainly $\overline{D} \subseteq X$, so let $x \in X$. Every basis element containing x intersects D, so $x \in \overline{D}$. □

1.31 Section 31: The Separation Axioms

Definitions: regular, normal
Main Idea: Regular spaces can separate points from closed sets. Normal spaces can separate closed sets from each other.

Definition 1.31.1. Let X be a space and suppose finite sets are closed in X. We say that X is **regular** if for all closed sets $C \subseteq X$ and $x \in X \setminus C$, there exist disjoint open sets U and V such that $C \subseteq U$ and $x \in V$.

Note that since finite sets, and in particular singletons, are closed, regular implies Hausdorff (recall **Lemma 1.26.9**). A regular space is one where we can separate points and closed sets.

Definition 1.31.2. We say that X is **normal** if for all disjoint closed sets $A, B \subseteq X$, there exist disjoint open sets $U, V \subseteq X$ such that $A \subseteq U$ and $B \subseteq V$.

Note that normal implies regular, since singletons are closed. A normal space is one where we can separate closed sets.

We now list some examples, with proofs to come:

Example 1.31.3. Metric spaces are normal (**Theorem 1.32.2**).

Example 1.31.4. Compact Hausdorff spaces are normal (**Theorem 1.32.3**).

Example 1.31.5. $\mathbf{R}$ with the K-topology is Hausdorff, but not regular. Recall the K-topology (**Definition 1.13.8**) where $K = \left\{\frac{1}{n}\right\}$ and a basis is either (a, b) or $(a', b') \setminus K$.[4]

Having the basis elements (a, b) is enough to show that $\mathbf{R}_K$ is Hausdorff. However, in the K topology, K is a closed set, and there does not exist disjoint open sets containing 0 and K. Thus $\mathbf{R}_K$ is not regular.

Lemma 1.31.6. *Suppose finite sets are closed in X. Then*

1. *X is regular if and only if for all $x \in X$ and neighborhoods $U \subseteq X$ of x, there exists a neighborhood $V \subseteq X$ of x such that $\overline{V} \subseteq U$.*

2. *X is normal if and only if for all closed sets $A \subseteq X$ and open sets $U \subseteq X$ such that $A \subseteq U$, there exists an open set $V \subseteq X$ such that $A \subseteq V$ and $\overline{V} \subseteq U$.*

Proof. We mirror the lemma in proof:

1. First, assume X is regular. Then fix $x \in X$ and $U \subseteq X$ a neighborhood of x. We wish to show there exists a neighborhood of x $V \subseteq X$ such that $\overline{V} \subseteq U$.

   So $X \setminus U$ is closed and does not contain x. As X is regular, there exist disjoint open sets $V, W \subseteq X$ such that $x \in V$ and $X \setminus U \subseteq W$. Then $V \subseteq X \setminus W$, and as $X \setminus W$ is closed, $\overline{V} \subseteq X \setminus W \subseteq U$, as desired.

   Now, fix an $x \in X$ and a closed subset $A \subseteq X$ such that $x \notin A$. We want to separate x and A, and thus X is regular. Well, $U = X \setminus A$ is an open neighborhood of x, so thus by hypothesis there exists a neighborhood $V \subseteq X$ of x such that $\overline{V} \subseteq U$. Then V and $X \setminus \overline{V}$ are disjoint open sets containing x and A, as desired, so X is regular.

[4] We see that in the K-topology, $\frac{1}{n}$ does not converge!

2. It turns out that the proof of 2. is identical in both directions, replacing x with a closed set B.

□

Theorem 1.31.7. *Subspaces and products of Hausdorff spaces are Hausdorff.*

Proof. Let X be Hausdorff and Y a subspace of X. Let $x, y \in Y$, distinct, and we wish to show we can separate them with open sets. As X is Hausdorff, there exist U and V disjoint that do the trick in X. Then $U \cap Y$ and $V \cap Y$ do the trick in the subspace.

Now let X_α be Hausdorff for all α. Let $\overline{x}, \overline{y} \in \prod X_\alpha$, distinct, and we wish to show we can separate them with open sets. As $\overline{x} \neq \overline{y}$, there exists β such that $\overline{x}(\beta) \neq \overline{y}(\beta) \in X_\beta$. This implies there exist disjoint open sets $U, V \subseteq X_\beta$ that separate $\overline{x}(\beta)$ and $\overline{y}(\beta)$. Then $\overline{x} \in {\pi_\beta}^{-1}(U)$ and $\overline{y} \in {\pi_\beta}^{-1}(V)$, which are disjoint and open by continuity of projection. □

Theorem 1.31.8. *Subspaces and products of regular spaces are regular.*

Proof. Let X be regular and Y a subspace of X. As finite sets are closed in X, they are also closed in Y. Let $A \subseteq Y$ be closed and $y \in Y \setminus A$. Then $A = Y \cap \overline{A}$, where $\overline{A}$ is the closure in X. Therefore $y \notin \overline{A}$, so since X is regular, there exist disjoint open sets $U, V \subseteq X$ such that $\overline{A} \subseteq U$ and $y \in V$. Then $U \cap Y$ and $V \cap Y$ satisfy $A \subseteq U \cap Y$ and $y \in V \cap Y$.

Now let X_α be regular for all α, and call $X = \prod X_\alpha$. Since regular implies Hausdorff, and the product of Hausdorff spaces is Hausdorff by **Theorem 1.31.7** above, X is Hausdorff, and hence finite sets in X are closed. Then take $\overline{x} \in X$ and U a neighborhood of $\overline{x}$. Then there exists an open set $\prod U_\alpha$ such that $\overline{x} \in \prod U_\alpha \subseteq U$ and $U_\alpha \neq X_\alpha$ for finitely many α. Then for each α such that $U_\alpha \neq X_\alpha$, choose a neighborhood V_α of $\overline{x}(\alpha)$ such that $\overline{V_\alpha} \subseteq U_\alpha$, as X_α is regular and using **Lemma 1.31.6**. For the other αs, set $V_\alpha = U_\alpha = X_\alpha$. Then $\overline{x} \in \prod V_\alpha$ and $\overline{\prod V_\alpha} = \prod \overline{V_\alpha} \subseteq \prod U_\alpha \subseteq U$. Thus X is regular, again using **Lemma 1.31.6**. □

Are subspaces and products of normal spaces normal? The answer is no.

1.32 Section 32: Normal Spaces

Definitions:
Main Idea: Normal spaces are pretty nice. Here's a few theorems above conditions you can apply to a space to get normality.

Theorem 1.32.1. *If X is second countable and regular, then X is normal.*

Proof. Let X be regular and let $\mathcal{B}$ be a countable basis for X. We need to show that we can separate closed sets in X. Let $C, D \subseteq X$ be disjoint closed sets. Since X is regular, for all $c \in C$, considering the neighborhood $X \setminus D$ of c, there exists a neighborhood U of c such that $\overline{U} \subseteq X \setminus D$ (this is once again **Lemma 1.31.6**).

Hence, we can cover C by $\{U_n\} \subseteq \mathcal{B}$ such that $\overline{U_n} \cap D = \emptyset$ for all n. Likewise, we can cover D by $\{V_n\} \subseteq \mathcal{B}$ where $\overline{V_n} \cap C = \emptyset$. We'd like to say that $\bigcup U_n$ and $\bigcup V_n$ do the trick and separate C and D, but there is a problem: some of the U_ns may intersect some of the V_ns, and hence the open sets aren't disjoint. To fix this, we need to remove any overlap of $\bigcup U_n$ and $\bigcup V_n$, but to maintain openness, we only remove closures.

So, to that end, define ${U_n}' = U_n \setminus \bigcup_{i=1}^{n} \overline{V_i}$, and define ${V_n}' = V_n \setminus \bigcup_{i=1}^{n} \overline{U_i}$. Then set $U' = \bigcup {U_n}'$ and $V' = \bigcup {V_n}'$. We claim that U' and V' do it; that $C \subseteq U'$, $D \subseteq V'$, and $U' \cap V' = \emptyset$.

To see this, since $\overline{U_n} \cap D = \emptyset$ and $\overline{V_n} \cap C = \emptyset$, we have that $C \subseteq U'$ and $D \subseteq V'$.

Now we show if $x \in U'$, $x \notin V'$, and vice versa. Let $x \in {U_n}'$. Then $x \notin \bigcup_{i=1}^{n} \overline{V_i}$, thus $x \notin {V_i}'$ for $1 \leq i \leq n$. Therefore ${U_n}' \cap {V_m}' = \emptyset$ if $m \leq n$. By a symmetric argument, ${U_n}' \cap {V_m}' = \emptyset$ if $m \geq n$. Therefore $U' \cap V' = \emptyset$, and X is normal, as desired. □

Theorem 1.32.2. *If X is metrizable, then X is normal.*

Proof. Suppose (X, d) is a metric space and $A, B \subseteq X$ are disjoint closed sets. Then we attempt to separate A and B with disjoint open sets.

For all $a \in A$, there exists $\varepsilon_a > 0$ such that $B_d(a, \varepsilon_a) \cap B = \emptyset$. Similarly, for all $b \in B$, there exists $\varepsilon_b > 0$ such that $B_d(b, \varepsilon_b) \cap A = \emptyset$. Then let $U = \bigcup_{a \in A} B_d\left(a, \frac{\varepsilon_a}{2}\right)$ and let $V = \bigcup_{b \in B} B_d\left(b, \frac{\varepsilon_b}{2}\right)$. We claim that U and V do it.

By construction $A \subseteq U$, $B \subseteq V$, and U and V are open. We only need to show U and V are disjoint. Suppose instead that $z \in U \cap V$. Then there exists $a \in A$ and $b \in B$ such that $z \in B_d\left(a, \frac{\varepsilon_a}{2}\right) \cap B_d\left(b, \frac{\varepsilon_b}{2}\right)$. But then $d(a, b) \leq d(a, z) + d(z, b) \leq \frac{1}{2}(\varepsilon_a + \varepsilon_b) \leq \max\{\varepsilon_a, \varepsilon_b\}$. Depending on which is the max, this shows that either $b \in B_d(a, \varepsilon_a)$, or $a \in B_d(b, \varepsilon_b)$, in either case a contradiction. Thus $U \cap V = \emptyset$. □

Theorem 1.32.3. *If X is compact and Hausdorff, then X is normal.*

Proof. Suppose X is a compact Hausdorff space and $A, B \subseteq X$ are disjoint closed sets. Once again, we try to separate A and B with disjoint open sets.

As X is compact, A and B are compact. Furthermore, we have already seen in **Lemma 1.26.9** that compact Hausdorff implies regular. Hence for all $a \in A$, there exists U_a, V_a open such that $a \in U_a$, $B \subseteq V_a$, and $U_a \cap V_a = \emptyset$. As A is compact, there exist finitely many $U_{a_1}, ..., U_{a_n}$ such that $A \subseteq \bigcup_{i=1}^{n} U_{a_i} = U$. Then let $V = \bigcap_{i=1}^{n} V_{a_i}$. We claim this works. See that $B \subseteq V$, and $U \cap V = \emptyset$. □

1.33 Section 33: Urysohn's Lemma

Definitions:
Main Idea: Urysohn's Lemma is the first theorem that doesn't really follow from definitions. Tricky!

Lemma 1.33.1 (**Urysohn's Lemma**). *Let X be a normal space and let $A, B \subseteq X$ be disjoint closed subsets. Then there exists a continuous function $f : X \to [0,1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.*[5]

Proof. Let $P = \mathbf{Q} \cap [0,1]$, and enumerate P; i.e., since P is countable, let $P = \{p_0 = 0, p_1 = 1, p_2, p_3, ...\}$, where we've specified the zeroth and first entries are 0 and 1, and the rest are any enumeration you like. Our goal will be to build a collection of open sets $\{U_p\}_{p \in P}$ where $\overline{U_p} \subseteq U_q$ if $p < q$. Then we will define $f(x) = \inf\{p \mid x \in U_p\}$, and show that f works.

So first, let $U_1 = X \setminus B$. Then this is an open set containing A. Since X is normal, there exists U_0 such that $A \subseteq U_0$, $U_0 \subseteq X$ is open, and $\overline{U_0} \subseteq U_1$. This is just **Lemma 1.31.6**.

That was the base case of constructing $\{U_p\}$ that does as we claim. We now move to the inductive step.

Now let $P_n = \{p_0, p_1, p_2, ..., p_n\}$, and suppose that there exists a collection of open sets $\{U_{p_i}\}_{i=0}^{n}$ such that $\overline{U_{p_i}} \subseteq U_{p_j}$ if $p_i < p_j$. Then there exist k and l such that $0 \le k \le n$ and $0 \le l \le n$ where p_k is the immediate predecessor of p_{n+1} and p_l is the immediate successor; i.e., P_{n+1}, in order, is $0 = p_0 < p_i < ... < p_k < p_{n+1} < p_l < ... < p_1 = 1$. In other words, we're squeezing p_{n+1} into the set in order.

Now, applying normality to $\overline{U_{p_k}} \subseteq U_{p_l}$, there exists $U_{p_{n+1}} \subseteq X$ open such that $\overline{U_{p_k}} \subseteq U_{p_{n+1}}$ and $\overline{U_{p_{n+1}}} \subseteq U_{p_l}$. Then induction has it; there exists a collection $\{U_p\}_{p \in P}$, as desired.

We can extend this construction to all of $\mathbf{Q}$ simply by

$$U_p = \begin{cases} \emptyset & \text{if } p < 0, \\ X & \text{if } p > 1. \end{cases}$$

This collection still satisfies $\overline{U_p} \subseteq U_q$ if $p < q$.

Now, we define $f : X \to [0,1]$ as described already; let $f(x) = \inf\{p \mid x \in U_p\}$. We need to show that f is continuous and that $f(A) = \{0\}$ and $f(B) = \{1\}$. We can do the latter quickly; if $x \in A$, then $x \in U_0$ and $x \notin U_p$ for $p < 0$. Therefore $f(x) = 0$. And if $x \in B$, then $x \in U_p$ for all $p > 1$ and $x \notin U_p$ for $p \le 1$. Therefore $f(x) = 1$.

[5]Note that if for all $A, B \subseteq X$ disjoint closed sets, there exists such an $f : X \to [0,1]$, then the sets $U = f^{-1}\left([0, \frac{1}{2})\right)$ and $V = f^{-1}\left((\frac{1}{2}, 1]\right)$ are the disjoint open sets containing A and B respectively need to confirm that X is normal. Therefore, we get the other direction relatively freely.

Note also that if $x \in \overline{U_r}$, then $f(x) \leq r$, since $x \in \overline{U_r} \subseteq U_s$ for all $s > r$, so $f(x) = \inf\{p \mid x \in U_p\} \leq r$.

Note further that if $x \notin U_r$, then $f(x) \geq r$, since $U_s \subseteq U_r$ for $s < r$ implies $x \notin U_s$, so $f(x) = \inf\{p \mid x \in U_p\} \geq r$.

All that remains is that we show that f is continuous; this completes the proof. We show the preimage of open sets is open. Suppose $f(x) \in (c, d)$. Then there exist $p, q \in \mathbf{Q}$ such that $c < p < f(x) < q < d$. Then take $U = U_q \setminus \overline{U_p}$. By our two notes, $f(U) \subseteq [p, q] \subseteq (c, d)$. We need to show $x \in U$, so suppose it isn't. Then either $x \notin U_q$, so $f(x) \geq q$, which is a contradiction, or $x \in \overline{U_p}$, so $f(x) \leq p$, again a contradiction. In either case, $x \in U$, and therefore f is continuous, as we wished to show. □

1.34 Section 34: Urysohn's Metrization Theorem

Definitions: embedding
Main Idea: Urysohn's Metrization Theorem says that second countable regular spaces are metrizable, via embedding them into $\mathbf{R}^J$. The specific embedding is interesting, but the statement of the theorem is probably more important.

Definition 1.34.1. An **embedding** is a map that is a homeomorphism onto its image. In other words, if $f : X \to Y$ is injective and continuous, f is an **embedding** if $X \cong_f f(X)$ (where $f(X) \subseteq Y$ has the subspace topology).

Theorem 1.34.2. *Suppose finite sets are closed in X. Suppose $\{f_\alpha : X \to \boldsymbol{R}\}_{\alpha \in J}$ is a collection of continuous functions such that for all $x_0 \in X$ and neighborhoods $U \subseteq X$ of x_0, there exists $\alpha_0 \in J$ for which $f_{\alpha_0}(x_0) > 0$ and $f_{\alpha_0}(X \setminus U) = 0$. Then $F(x) = (f_\alpha(x))_{\alpha \in J} \in \boldsymbol{R}^J$ is an embedding.*

Proof. We show that F is a homeomorphism by showing F is continuous, injective, and F^{-1} is continuous (which we do by showing F is an open map). Note that injectiveness is enough, since we're only showing a homeomorphism on the image of F.

First, F is continuous as each f_α is.

Next, to see that F is injective, let $x, y \in X$ such that $x \neq y$. We wish to show $F(x) \neq F(y)$. See that $X \setminus \{y\}$ is a neighborhood of x. Thus by hypothesis, there exists $\alpha_0 \in J$ such that $f_{\alpha_0}(x) > 0$ and $f_{\alpha_0}(y) = 0$. Since $f_{\alpha_0}(x) \neq f_{\alpha_0}(y)$, therefore $F(x) \neq F(y)$. Thus F is injective.

Now we show that F is an open map. Let $U \subseteq X$ be open. We show that $F(U)$ is open in $Z = F(X) \subseteq \mathbf{R}^J$. Fix $z_0 \in F(U)$ and let $x_0 \in X$ be such that $F(x_0) = z_0$. Then U is a neighborhood of x_0, and by hypothesis once more there exists $\alpha_0 \in J$ such that $f_{\alpha_0}(x_0) > 0$ and $f_{\alpha_0}(X \setminus U) = 0$. Now let $V = {\pi_{\alpha_0}}^{-1}((0, \infty))$. In other words, V is the set of elements with α_0-coordinate greater than 0. This is open in $\mathbf{R}^J$. Then $W = V \cap Z$ is a neighborhood of z_0. We claim that $W \subseteq F(U)$. Indeed, if $z \in Z \setminus F(U) = F(X \setminus U)$, then $z = F(y)$ for some $y \in X \setminus U$. Then $\pi_{\alpha_0}(z) = f_{\alpha_0}(y) = 0$ as $y \in X \setminus U$. Thus, $z \notin W$. So

$W \subseteq F(U)$, and F is an open map. Therefore, F is a homeomorphism onto its image, i.e., an embedding. $\square$

Theorem 1.34.3 (**Urysohn's Metrization Theorem**). *If X is second countable and regular, then X is metrizable.*

The idea here is to embed X into a metric space.

Proof. Let X be second countable and regular. We will define an embedding $X \to \mathbf{R}^{\mathbf{N}}$. Note that the $\mathbf{N}$ is a result of second countability, and $\mathbf{R}^{\mathbf{N}}$ is metrizable in the product topology, as we've seen the metric $D(\overline{x}, \overline{y}) = \sup\left\{\frac{\overline{d}(x_n, y_n)}{n} \mid n \in \mathbf{N}\right\}$ (where $\overline{d}$ is the standard bounded metric) in **Definition 1.20.15**.

We know that X second countable and regular implies that X is normal by **Theorem 1.32.1**, so we can use **Urysohn's Lemma 1.33.1**.

Let $\{B_n\}_{n\in\mathbf{N}}$ be a countable basis of X. For all $m, n \in \mathbf{N}$ such that $\overline{B_n} \subseteq B_m$, by **Urysohn's Lemma 1.33.1**, there exists a continuous function $g_{m,n} : X \to [0, 1]$ such that $g_{m,n}(\overline{B_n}) = 1$ and $g_{m,n}(X \setminus B_m) = 0$. Then we claim that $\{g_{m,n} : X \to [0, 1]\}$ satisfies the hypotheses of **Theorem 1.34.2**, and therefore $G(x) = (g_{m,n}(x))_{m,n\in\mathbf{N}} \in \mathbf{R}^{\mathbf{N}}$ is an embedding.

To see this, fix $x_0 \in X$ and let $U \subseteq X$ be a neighborhood of x_0. There exists a basis element B_m such that $x_0 \in B_m \subseteq U$. Since X is regular, there exists a neighborhood V of x_0 such that $\overline{V} \subseteq B_m$. Then there exists B_n such that $x_0 \in B_n$ and $\overline{B_n} \subseteq \overline{V} B_m$. Then $g_{m,n}(x_0) = 1 > 0$ and as $X \setminus U \subseteq X \setminus B_m$, we have $g_{m,n}(X \setminus U) = 0$. This is enough; therefore G is an embedding. $\square$

Note that we cannot weaken the hypotheses of **Urysohn's Metrization Theorem 1.34.3** any further. For example, we need X to be regular, not just Hausdorff:

Example 1.34.4. Consider the space $\mathbf{R}_K$ (from **Definition 1.13.8**). We saw in **Example 1.31.5** that $\mathbf{R}_K$ is not regular, since we cannot separate 0 from the closed set K. Since $\mathbf{R}_K$ is not regular, it is not metrizable (since metrizable implies normal implies regular, see **Theorem 1.32.2** and **Definition 1.31.2**). But in **Example 1.31.5** we saw that $\mathbf{R}_K$ is Hausdorff. In fact, it is also second countable: a countable basis is the collection of (q, r) and $(q, r) \setminus K$, where $q, r \in \mathbf{Q}$.

1.35 Section 35: The Tietze Extension Theorem

Definitions:
Main Idea: The Tietze Extension Theorem says in normal spaces, we can extend functions on closed subspaces to functions on the whole space.

Theorem 1.35.1 (**The Tietze Extension Theorem**). *Let X be normal, and let $A \subseteq X$ be closed.*

1. *Any continuous function $f : A \to [a, b]$ can be extended to X; i.e., there exists a continuous $F : X \to [a, b]$ where $F|_A = f$.*

2. *Any continuous function $f : A \to \boldsymbol{R}$ can be extended to X; i.e., there exists a continuous function $F : X \to \boldsymbol{R}$ where $F|_A = f$.*

Proof. In this proof, we'll build a sequence of functions on X that approximate f on A. Under a limit, we'll have it. The key tool in this proof is this:

> Suppose we have a continuous $f : A \to [-r, r]$. Break $[-r, r]$ into $\left[-r, -\frac{r}{3}\right] \cup \left[-\frac{r}{3}, \frac{r}{3}\right] \cup \left[\frac{r}{3}, r\right]$. Then let $B = f^{-1}\left(\left[-r, -\frac{r}{3}\right]\right)$ and let $C = f^{-1}\left(\left[\frac{r}{3}, r\right]\right)$. Then by continuity of f, B and C are closed and disjoint in X. By **Urysohn's Lemma 1.33.1**, there exists $g : X \to \left[-\frac{r}{3}, \frac{r}{3}\right]$ such that $g(B) = -\frac{r}{3}$ and $g(C) = \frac{r}{3}$. In other words, g is the function that caps off f with a constant piece when it goes above $\frac{r}{3}$ or below $-\frac{r}{3}$ and is defined on all of X. Notice then that for every $a \in A$, $|f(a) - g(a)| \leq \frac{2r}{3}$; i.e., both $f(a)$ and $g(a)$ lie on the same interval of width $\frac{2r}{3}$.

So now let's prove the theorem. We start with the first part:

1. Let $f : A \to [-1, 1]$; we wish to show we can extend f to X. By the key tool, there exists $g_1 : X \to \left[-\frac{1}{3}, \frac{1}{3}\right]$ where $|f(a) - g_1(a)| \leq \frac{2}{3}$ for $a \in A$. Now consider $f - g_1 : A \to \left[-\frac{2}{3}, \frac{2}{3}\right]$. By the key tool, there exists $g_2 : X \to \left[-\frac{2}{9}, \frac{2}{9}\right]$ where $|f(a) - g_1(a) - g_2(a)| \leq \left(\frac{2}{3}\right)^2$ for $a \in A$. Continue inductively in this fashion to build $g_n : X \to \left[-\frac{1}{2}\left(\frac{2}{3}\right)^n, \frac{1}{2}\left(\frac{2}{3}\right)^n\right]$ where $|f(a) - g_1(a) - ... - g_n(a)| \leq \left(\frac{2}{3}\right)^n$ for $a \in A$.

   Now, set $g(x) = \displaystyle\sum_{n=1}^{\infty} g_n(x)$. As $|g_n(x)| \leq \frac{1}{3}\left(\frac{2}{3}\right)^{n-1}$, we have

   $$|g(x)| = \left|\sum_{n=1}^{\infty} g_n(x)\right| \leq \frac{1}{3}\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n.$$

   By the **Weierstrass M-test**, we have that $g : X \to [-1, 1]$ is well defined and continuous. We claim that g an extension of f. We have already seen that g is continuous and maps the correct domain and codomain. It just remains to be seen that $g|_A = f$.

   To see this, let $s_n(x) = \displaystyle\sum_{i=1}^{n} g_i(x)$, so that $s_n \to g$. If $a \in A$, then $|f(a) - s(a)| = |f(a) - g_1(a) - ... - g_n(a)| \leq \left(\frac{2}{3}\right)^n$. As $n \to \infty$, $s_n \to f$, so for $a \in A$, $g(a) = f(a)$, as desired.

2. Now let's prove the second part, where $f : A \to \mathbf{R}$. We can compose f with a homeomorphism so that we prove the theorem for $f_* : A \to (-1, 1) \subseteq [-1, 1]$. By part 1., we can extend f_* to a map $g : X \to [-1, 1]$, but we need the extension to map into the open interval $(-1, 1)$.

To that end, let $D = g^{-1}(\{-1,1\})$. Then D is closed, since it is the preimage of a closed set, and it is disjoint from A, since $g(A) = f(A) \subseteq (-1,1)$. Then by **Urysohn's Lemma 1.33.1**, there exists $h : X \to [0,1]$ where $h(\{0\}) = \{0\}$ and $h(A) = \{1\}$. Then let $F(x) = h(x) \cdot g(x)$; we claim this F works. We need to show F is continuous, $F|_A = f$, and that $F : X \to (-1,1)$.

First, F is continuous, since h and g are. Second, if $a \in A$, then $F(a) = h(a)g(a) = g(a) = f(a)$. Finally, if $x \notin D$, then $|F(x)| < 1$, and if $x \in D$, then $F(x) = 0 < 1$. Therefore $F(x) \in (-1,1)$, and we have done it. $\square$

1.36 Section 36: Embeddings of Manifolds

Definitions: manifold, surface, support, partition of unity
Main Idea: Getting comfortable with manifolds; we will explore them much deeper in the future and in the second semester. That every m-manifold can be embedded into $\mathbf{R}^n$ is important, but the proof is not.

Definition 1.36.1. A second countable Hausdorff space X such that every point has a neighborhood homeomorphic to an open set in $\mathbf{R}^m$ is called an ***m*-manifold**. We can also say that an m-manifold is second countable, Hausdorff, and locally Euclidean.

Example 1.36.2. Clearly $\mathbf{R}^m$ is a manifold.

Example 1.36.3. $S^1 = \{(x,y) \in \mathbf{R}^2 \mid x^2 + y^2 = 1\}$ is a 1-manifold. It's obvious with a year of topology baggage, but let's show it explicitly. Consider the four charts $U_+ = \{(1,\theta) \mid 0 < \theta < \pi\}$, $U_- = \{(1,\theta) \mid \pi < \theta < 2\pi\}$, $V_+ = \{(1,\theta) \mid -\frac{\pi}{2} < \theta < \frac{\pi}{2}\}$, and $V_- = \{(1,\theta) \mid \frac{\pi}{2} < \theta < \frac{3\pi}{2}\}$. Then, for example, $\varphi : U_+ \to (0,\pi)$, $\varphi((1,\theta)) = \theta$ is a homeomorphism. Similarly, each of $U_\pm, V_\pm$ is homeomorphic to $\mathbf{R}$.

Example 1.36.4. Now consider $S^n = \left\{(x_0, ..., x_n) \in \mathbf{R}^{n+1} \mid \sum {x_i}^2 = 1\right\}$. S^n can be covered by $2(n+1)$ charts; e.g., using cartesian coordinates this time,

$$U_{i,+} = \left\{(x_0, ..., x_n) \mid \sum_{j \neq i} {x_j}^2 < 1, x_i = \sqrt{1 - \sum_{j \neq i} {x_j}^2}\right\}.$$

Then $\varphi_{i,+} : U_{i,+} \to B_d(\overline{0}, 1) \subseteq \mathbf{R}^n$ by dropping the ith coordinate is a homeomorphism. Thus S^n is an n-manifold.

Example 1.36.5. Let's see something that is not a manifold. Consider $X = \mathbf{R} \cup \{*\}$, the line with two origins. Neighborhoods of $\{*\}$ are $U \setminus \{0\} \cup \{*\}$, where U is a neighborhood of $\{0\}$. The line with two origins is not Hausdorff, since $\{0\}$ and $\{*\}$ cannot be separated. Thus it is not a manifold.

Example 1.36.6. Another non-example is $(-1,1)\times\mathbf{R}$ where we give $\mathbf{R}$ the discrete topology. We can think of this space as an uncountable union of intervals. This space isn't second countable, so it is not a manifold.

Proposition 1.36.7. *Any connected 1-manifold is homeomorphic to either $\boldsymbol{R}$ or S^1.*

Definition 1.36.8. A 2-manifold is called a **surface**.

Example 1.36.9. Some examples of manifolds which are orientable (**Definition ??**) are S^2, the torus $S^1 \times S^1$, and surfaces of genus 2, 3, and so on.

Example 1.36.10. Let's construct a non-orientable manifold. This is the real projective plane $\mathbf{RP}^2$.

We begin by defining an equivalence relation on $S^2 \subseteq \mathbf{R}^3$ by $\overline{x} \sim -\overline{x}$. We can see that every equivalence class has a representative where the z-coordinate is nonnegative. This representative is unique except on the equator, where $z = 0$. Thus we can think of ${S^2}/{\sim}$ as the upper hemisphere, homeomorphic to a closed disk. Then we identify antipodal points on the boundary. This space doesn't exist in $\mathbf{R}^3$. A picture was drawn that isn't very helpful. It's easier to think of $\mathbf{RP}^2$ as a quotient space, an identified polygon.

Definition 1.36.11. The **support** of a map $\varphi : X \to \mathbf{R}$ is defined to be $\operatorname{supp}(\varphi) = \overline{\varphi^{-1}(\mathbf{R}\setminus\{0\})}$. The support of a map is the closure of its nonzero image.

Definition 1.36.12. Suppose $\{U_1, ..., U_n\}$ is a cover of a space X. A family of continuous functions $\{\varphi_i : X \to \mathbf{R}\}_{i=1}^n$ is a **partition of unity** dominated by (or subordinate to) $\{U_i\}$ if $\operatorname{supp}(\varphi_i) = \overline{{\varphi_i}^{-1}(\mathbf{R}\setminus\{0\})} \subseteq U_i$ and for all $x \in X$, $\displaystyle\sum_{i=1}^n \varphi_i(x) = 1$.

Theorem 1.36.13. *If $\{U_i\}_{i=1}^n$ is a cover of a normal space, then there exists a partition of unity dominated by the cover $\{U_i\}_{i=1}^n$.*

Proof. We give a bird's eye view of the proof. Since X is normal, we can find covers $\{W_i\}$ and $\{V_i\}$ where $\overline{W_i} \subseteq V_i$ and $\overline{V_i} \subseteq U_i$. Then **Urysohn's Lemma 1.33.1** applied to $\overline{W_i}$ and $X \setminus V_i$ gives continuous functions $f_i : X \to [0,1]$ such that $f_i(\overline{W_i}) = \{1\}$ and $f_i(X \setminus V_i) = \{0\}$. This actually will give you the functions needed, but they don't sum to 1. However, since there's only finitely many, just normalize them so that they do. $\square$

Theorem 1.36.14. *If X is a compact m-manifold, then X can be embedded into $\boldsymbol{R}^n$.*

Proof. As X is compact, cover X by $\{U_i\}_{i=1}^n$. Let $g_i : U_i \to \mathbf{R}^m$ be embeddings. There are finitely many g_is by the compactness of X. Furthermore, X is normal since it is compact and Hausdorff. Let $\{\varphi_i : X \to \mathbf{R}\}$ be the corresponding partition of unity. Then define $h_i : X \to \mathbf{R}^m$ by

$$h_i(x) = \begin{cases} \varphi_i(x)g_i(x) & \text{if } x \in U_i, \\ \bar{0} & \text{if } x \in X \setminus \operatorname{supp}(\varphi_i). \end{cases}$$

Then all h_i are continuous, by the **Pasting Lemma 1.18.9**.

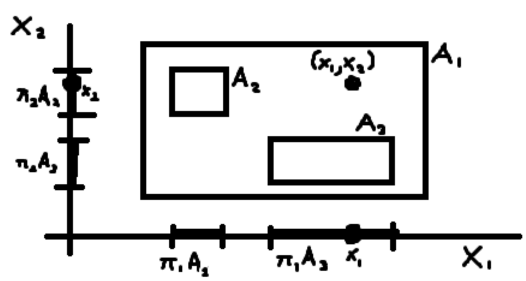
Define $f : X \to \mathbf{R}^n \times (\mathbf{R}^m)^n$ by $f(x) = (\varphi_1(x), ..., \varphi_n(x), h_1(x), ..., h_n(x))$. We claim that f is the embedding of the m-manifold. Since X is compact and Hausdorff, it suffices to show f is injective to be a homeomorphism. To see this, suppose $f(x) = f(y)$. Then there exists i such that $\varphi_i(x) = \varphi_i(y) > 0$. This implies that $x, y \in \operatorname{supp}(\varphi_i) \subseteq U_i$.Thus $\varphi_i(x)g_i(x) = h_i(x) = h_i(y) = \varphi_i(y)g_i(y)$, and since $\varphi_i(x) > 0$, $g_i(x) = g_i(y)$. Since g_i is an embedding, g is injective, and therefore $x = y$, as desired. $\square$

1.37 Section 37: The Tychonoff Theorem

Definitions:
Main Idea: Products of compact spaces are compact.

Before we give the **Tychonoff Theorem 1.37.3**, we outline some idea of the proof. First, note that we will use the closed set formulation of compactness, i.e., the finite intersection property (**Definition 1.26.14**). The main idea is this:

Let X_1 and X_2 be compact. Suppose $\mathcal{A}$ is a collection of subsets of $X_1 \times X_2$ with the finite intersection property. Then consider the collections $\mathcal{A}_1 = \{\overline{\pi_1(A)} \mid A \in \mathcal{A}\}$ and $\mathcal{A}_2 = \{\overline{\pi_2(A)} \mid A \in \mathcal{A}\}$. By construction, $\mathcal{A}_1$ is a collection in X_1 and $\mathcal{A}_2$ is a collection in X_2. Each has the finite intersection property. As each X_i is compact, there exists x_i such that $x_i \in \bigcap_{A \in \mathcal{A}_i} A$. We ask: is $(x_1, x_2) \in \bigcap_{A \in \mathcal{A}} A$? The answer is no, not necessarily. Not only could (x_1, x_2) miss at least one A_i, it could miss all of them! To see this, consider the following picture.



Therefore, the main idea will be to enlarge $\mathcal{A}$ so that this will work. The more sets we add, the more restrictive the intersection will become.

Lemma 1.37.1. *Let X be a set and $\mathcal{A}$ a collection of subsets with the finite intersection property. Then there exists a collection $\mathcal{D}$ such that*

1. *$\mathcal{A} \subseteq \mathcal{D}$,*
2. *$\mathcal{D}$ has the finite intersection property, and*
3. *if $\mathcal{D} \subseteq \mathcal{D}'$ and $\mathcal{D}'$ has the finite intersection property, then $\mathcal{D} = \mathcal{D}'$.*

We say that $\mathcal{D}$ is maximal with respect to the finite intersection property.

Proof. We use **Zorn's Lemma**: As long as simply ordered subsets have an upper bound, then maximal elements exist.

Let $\mathcal{X}$ be the collection of subsets of X with the finite intersection property that contain $\mathcal{A}$. Let $\mathcal{Y} \subseteq \mathcal{X}$ be a simply ordered subset, and let $\mathcal{Z} = \bigcup_{Y \in \mathcal{Y}} Y$, where each Y is a collection of subsets in X. We claim this is the collection we seek. It suffices to show that $\mathcal{Z}$ has the finite intersection property, since $\mathcal{Z}$ is an upper bound to $\mathcal{Y}$ and $\mathcal{A} \subseteq \mathcal{Z}$.

Let $C_1, ..., C_n \in \mathcal{Z}$. As $\mathcal{Y}$ is simply ordered, there exists $Y \in \mathcal{Y}$ such that $C_1, ..., C_n \in Y$. As Y has the finite intersection property, $C_1 \cap ... \cap C_n \neq \emptyset$. Hence $\mathcal{Z}$ has the finite intersection property. $\square$

Lemma 1.37.2. *Suppose $\mathcal{D}$ is maximal with respect to the finite intersection property. Then:*

1. *If $C_1, ...C_n \in \mathcal{D}$, then $C_1 \cap ... \cap C_n \in \mathcal{D}$.*
2. *If $A \cap C \neq \emptyset$ for all $C \in \mathcal{D}$, then $A \in \mathcal{D}$.*

Proof. We prove each at a time:

1. Let $B = C_1 \cap ... \cap C_n$. We'll show that $\mathcal{D} \cup \{B\}$ has the finite intersection property, and the result follows by maximality of $\mathcal{D}$.

   Given $D_1, ..., D_m \in \mathcal{D}$, then $B \cap D_1 \cap ... \cap D_m = (C_1 \cap ... \cap C_n) \cap D_1 \cap ... \cap D_m \neq \emptyset$, since it is a finite collection from $\mathcal{D}$. Thus $\mathcal{D} \cup \{B\} = \mathcal{D}$.

2. As in 1., we show $\mathcal{D} \cup \{A\}$ has the finite intersection property. Let $C_1, ..., C_n \in \mathcal{D}$. Then $C = C_1 \cap ... \cap C_n \in \mathcal{D}$ by part 1. Then $A \cap C_1 \cap ... \cap C_n = A \cap C \neq \emptyset$. $\square$

Theorem 1.37.3 (**Tychonoff Theorem**). *If $\{X_\alpha\}_{\alpha \in J}$ is a family of compact spaces, then $X = \prod_{\alpha \in J} X_\alpha$ is compact (in the product topology).*

Proof. Let $\mathcal{A}$ be a collection of subsets of X with the finite intersection property. We'll show $\bigcap_{A \in \mathcal{A}} \overline{A} \neq \emptyset$ (**Theorem 1.26.15**).

By **Lemma 1.37.1**, let $\mathcal{D}$ be maximal with respect to the finite intersection property and with $\mathcal{A} \subseteq \mathcal{D}$. It suffices, therefore, to show that $\bigcap_{D\in\mathcal{D}} \overline{D} \neq \emptyset$, as we're taking the intersection over more sets, which can only shrink.

So, as each X_α is compact, there exists $x_\alpha \in \bigcap_{D\in\mathcal{D}} \overline{\pi_\alpha(D)}$. We claim that $\overline{x} = (x_\alpha)_{\alpha\in J} \in \bigcap_{D\in\mathcal{D}} \overline{D}$.

To see this, fix $\alpha \in J$. Then for all $U_\alpha \subseteq X_\alpha$ neighborhood of x_α, we have that $U_\alpha \cap \pi_\alpha(D) \neq \emptyset$ for all $D \in \mathcal{D}$, since $x_\alpha \in \overline{\pi_\alpha(D)}$. This implies that ${\pi_\alpha}^{-1}(U_\alpha) \cap D \neq \emptyset$ for all $D \in \mathcal{D}$.

By maximality of $\mathcal{D}$ we can use part 2. of **Lemma 1.37.2** to thus show that ${\pi_\alpha}^{-1}(U_\alpha) \in \mathcal{D}$ for all $\alpha \in J$ and for all neighborhoods U_α of x_α.

Then by part 1. of **Lemma 1.37.2**, for all $\alpha_1, ...\alpha_k \in J$, the set ${\pi_{\alpha_1}}^{-1}(U_{\alpha_1}) \cap ... \cap {\pi_{\alpha_k}}^{-1}(U_{\alpha_k}) \in \mathcal{D}$ for all neighborhoods U_{α_i} of x_{α_i}. But ${\pi_{\alpha_1}}^{-1}(U_{\alpha_1}) \cap ... \cap {\pi_{\alpha_k}}^{-1}(U_{\alpha_k})$ is a basis element for the product topology that contains $\overline{x}$. Hence every basis element of the product topology containing $\overline{x}$ is in $\mathcal{D}$, and hence intersects every $D \in \mathcal{D}$ nontrivially. There $\overline{x} \in \overline{D}$ for all $D \in \overline{D}$, and thus $\bigcap_{D\in\mathcal{D}} \overline{D} \neq \emptyset$, as desired. □

1.43 Section 43: Complete Metric Spaces

Definitions: Cauchy, complete, bounded, set of bounded functions, set of continuous functions, sup metric, completion
Main Idea: Completeness is not a topological invariant. Of importance are $\mathcal{B}(X,Y)$ and $\mathcal{C}(X,Y)$.

Definition 1.43.1. Let (X,d) be a metric space. A sequence is **Cauchy** if for every $\varepsilon > 0$, there exists $N > 0$ such that for all $m, n \geq N$, $d(x_n, x_m) < \varepsilon$.

Definition 1.43.2. (X,d) is called **complete** if every Cauchy sequence converges.

Note that if we consider the standard bounded metric $\overline{d}(x,y) = \min\{d(x,y), 1\}$, then (X,d) is complete if and only if $(X,\overline{d})$ is.

Lemma 1.43.3. *(X,d) is complete if and only if every Cauchy sequence has a convergent subsequence.*

Proof. If (X,d) is complete, then every Cauchy sequence converges, so take the subsequence of the sequence itself.

If every Cauchy sequence has a convergent subsequence, then let (x_i) be Cauchy and $(x_{i_k} \to x$ a convergent subsequence. Then for any $\varepsilon > 0$, there exists N such that for all $n, m \geq N$, $d(x_n, x_m) < \frac{\varepsilon}{2}$. Also, there exists M such that for all $k \geq M$, $d(x_{i_k}, x) < \frac{\varepsilon}{2}$. Then for $n, m \geq N, M$ there exists $i_k > M, N$ such that, by the triangle inequality, $d(x_n, x) \leq d(x_n, x_{i_k}) + d(x_{i_k}, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, so $(x_n) \to x$. □

Theorem 1.43.4. *The metric spaces $(\boldsymbol{R}^k, d)$ and $(\boldsymbol{R}^k, \rho)$ are complete.*

Recall that ρ is the square metric in **Definition 1.20.10**.

Proof. Let's first show that $(\mathbf{R}^k, \rho)$ is complete. Let (x_n) be a Cauchy sequence in $(\mathbf{R}^k, \rho)$; we wish to show it converges. We first claim that the set $\{x_n\}$ is bounded in $\mathbf{R}^k$.

To see this, there exists N such that for all $n, m \geq N$, $d(x_n, x_m) < \frac{1}{2}$, as (x_n) is Cauchy. Set $M = \max\{\rho(x_1, 0, ..., 0) + 1, \rho(x_2, 0, ..., 0) + 1, ..., \rho(x_N, 0, ..., 0) + 1\}$. Then $x_1, ..., x_N \in B_\rho(M, 0, ..., 0)$. Also, for $n \geq N$, $\rho(x_n, x_N) < \frac{1}{2}$, so $x_n \in B_\rho(M, 0, ...0)$. Therefore for all n, $x_n \in [-M, M]^k$, so $\{x_n\}$ is bounded.

Since $[-M, M]^n$ is compact and in a metric space, this is equivalent to sequential compactness (see **Definition 1.28.4** and **Theorem 1.28.5**), (x_n) has a convergent subsequence. Therefore, by **Lemma 1.43.3**, $(\mathbf{R}^k, \rho)$ is complete.

Now we show that $(\mathbf{R}^k, d)$ is complete. Note that a sequence is Cauchy in $(\mathbf{R}^k, d)$ if and only if it is Cauchy in $(\mathbf{R}^k, \rho)$, and a sequence converges in $(\mathbf{R}^k, d)$ if and only if it converges in $(\mathbf{R}^k, \rho)$. Therefore, by the first half of this proof, we are done. □

Lemma 1.43.5. *Let $X = \prod X_\alpha$ and $\pi_\alpha : X \to X_\alpha$ be the projection onto the α-coordinate. Then $(x_n) \to x \in X$ if and only if, for all α, $(\pi_\alpha(x_n)) \to \pi_\alpha(x) \in X_\alpha$.*

Proof. Let $(x_n) \to x$. Then for all α, π_α is continuous, so by **Theorem 1.21.7**, $(\pi_\alpha(x_n)) \to \pi_\alpha(x)$.

Now, assume for all α that $(\pi_\alpha(x_n)) \to \pi_\alpha(x)$. Then using all α, $\pi_\alpha(x)$ defines an $x \in X$, and for all n, $\pi_\alpha(x_n)$ defines an $x_n \in X$, by the **Axiom of Choice**. Simply choose x and x_n such that every αth coordinate works. We now claim that $(x_n) \to x$.

Now let $U \subseteq X$ be a neighborhood of x, and let $x \in B \subseteq U$ where B is a basis element. Then $B = \prod_{\alpha \in J} U_\alpha$. So in each coordinate α there exists N_α such that for all $n \geq N_\alpha$, $\pi_\alpha(x_n) \in U_\alpha$, by its convergence. Then set $N = \sup\{N_\alpha\}$. In the product topology, this will not be infinity, since for all but finitely many α we have $N_\alpha = 0$. Thus $N \in \mathbf{N}$, and $(x_n) \to x$, as desired. □

Theorem 1.43.6. *$\boldsymbol{R}^{\boldsymbol{N}}$ is complete.*

Note that a Cauchy sequence in $\mathbf{R}^{\mathbf{N}}$ is (x_n) where each $x_n = (x_{n_1}, x_{n_2}, ...)$. Note further that completeness *depends on the metric.* Thus our first goal will be to procure a metric D such that $(\mathbf{R}^{\mathbf{N}}, D)$ is complete.

Proof. Let $(\mathbf{R}, \overline{d})$ be $\mathbf{R}$ with the standard bounded metric, $\overline{d}(a, b) = \min\{|a - b|, 1\}$. Then recall the product metric, **Definition 1.20.15**, which makes a metric space $(\mathbf{R}^{\mathbf{N}}, D)$, where $D(\overline{x}, \overline{y}) = D((x_1, x_2, ...), (y_1, y_2, ...)) = \sup\left\{\frac{\overline{d}(x_i, y_i)}{i}\right\}$. By **Theorem 1.20.16**, D induces the product topology. We now claim that $(\mathbf{R}^{\mathbf{N}}, D)$ is complete.

Let (x_n) be Cauchy. Then for all i, $\overline{d}(\pi_i(x), \pi_i(y)) \leq iD(x, y)$. Since (x_n) is Cauchy, for all i, $\pi_i((x_n))$ is Cauchy. By **Lemma 1.43.5**, we are done. □

Example 1.43.7. Here are some spaces that are/are not complete:

1. $\mathbf{Q}$ is not complete.
2. $\mathbf{R}$ is complete.
3. $(0, 1)$ is not complete.
4. $(0, \infty)$ is not complete.

Note something very important! We have just said $\mathbf{R}$ is complete and $(0, \infty)$ is not. But $\mathbf{R} \cong (0, \infty)$ (one such homeomorphism is $\exp : \mathbf{R} \to (0, \infty)$, with inverse ln). Therefore, completeness is **not** a topological invariant, and our examples are a bit misleading, because this remark shows we can find metrics to make spaces complete (compose with homeomorphisms, for example).

Now recall **Definition 1.20.13**; the uniform metric is defined to be $\overline{\rho}(\overline{x}, \overline{y}) = \sup\{\overline{d}(x_\alpha, y_\alpha) \mid \alpha \in J\}$.

Theorem 1.43.8. *If (Y, d) is complete, then $(Y^J, \overline{\rho})$ is complete.*

Proof. Let (x_i) be a Cauchy sequence in Y^J; we need to show it converges. Recall from **Definition 1.19.1** that $Y^J = \{f : J \to Y\}$. Then we rename (x_i) to (f_n).

Fix $\alpha \in J$. Then $\overline{d}(f_n(\alpha), f_m(\alpha)) \leq \overline{\rho}(f_n, f_m)$. Thus for every α, $f_n(\alpha)$ is Cauchy. Since Y is complete, $(f_n(\alpha)) \to f(\alpha) \in Y$. Therefore we get a function $f : J \to Y$, or in other words, $f \in Y^J$. Then, we have that $\overline{\rho}(f_n, f) = \sup\limits_{\alpha \in J}\{\overline{d}(f_n(\alpha), f)\} < \varepsilon$ for n sufficiently large, and thus (f_n) converges. $\square$

In particular, considering the previous theorem, we can take $J = X$ a topological space. Then (Y, d) complete implies $(Y^X, \overline{\rho})$ is complete. We now take some time to talk about $Y^X = \{f : X \to Y\}$, a function space.

Definition 1.43.9. A function $f \in Y^X$ is called **bounded** if there exists $M > 0$ such that for all $x_1, x_2 \in X$, $d(f(x_1), f(x_2)) < M$.

Definition 1.43.10. Define $\mathcal{B}(X, Y) \subseteq Y^X$ to be the **set of bounded functions from X to Y**; i.e., $f \in \mathcal{B}(X, Y)$ if $f : X \to Y$ is bounded.

Definition 1.43.11. Define $\mathcal{C}(X, Y) \subseteq Y^X$ to be the **set of continuous functions from X to Y**; i.e., $f \in \mathcal{C}(X, Y)$ if $f : X \to Y$ is continuous.

Theorem 1.43.12. *Let (Y, d) be a metric space and let X be any topological space. Then $\mathcal{B}(X, Y)$ and $\mathcal{C}(X, Y)$ are closed in $(Y^X, \overline{\rho})$.*

Note that therefore, if (Y, d) is complete, then $(\mathcal{B}(X, Y), \overline{\rho})$ and $(\mathcal{C}(X, Y), \overline{\rho})$ are complete.

Proof. We start with showing $\mathcal{B}(X, Y)$ is closed. Let f be a limit point of $\mathcal{B}(X, Y)$; we show that $f \in \mathcal{B}(X, Y)$, i.e., that f is bounded. Since f is a limit point, there exists a sequence $(f_n) \subseteq \mathcal{B}(X, Y)$ such that $(f_n) \to f$. Then let N

be such that $\overline{\rho}(f_N, f) < \frac{1}{2}$. Thus $\overline{d}(f_N(x), f(x)) < \frac{1}{2}$, so $d(f_N(x), f(x)) < \frac{1}{2}$. Since $f_N \in \mathcal{B}(X, Y)$, let M be the bound on $f_N(X)$; i.e., $\text{diam}(f_N(X)) < M$. Then $\text{diam}(f(X)) < M + 1$, so f is bounded, as desired.

Now, we show $\mathcal{C}(X, Y)$ is closed. Let f be a limit point of $\mathcal{C}(X, Y)$; we show that $f \in \mathcal{C}(X, Y)$, i.e., that f is continuous. Since f is a limit point, there exists a sequence $(f_n) \subseteq \mathcal{C}(X, Y)$ such that $(f_n) \to f$. We show that the convergence is uniform (**Definition 1.21.8**), and therefore by **Theorem 1.21.9**, f is continuous.

To see this, let $\varepsilon > 0$. Then, since $(f_n) \to f$, there exists N such that if $n \geq N$, $\overline{\rho}(f, f_n) < \varepsilon$. Thus $\overline{d}(f_n(x), f(x)) < \varepsilon$, for all $x \in X$. Thus (f_n) converges to f uniformly, and we have that f is continuous, as desired. □

Definition 1.43.13. We define the sup**-metric on** $\mathcal{B}(X, Y)$ by, if $f, g \in \mathcal{B}(X, Y)$, $\rho(f, g) = \sup_{x \in X}\{d(f(x), g(x))\}$.

Lemma 1.43.14. *The* sup*-metric is a metric.*

Proof. Reflexive and symmetric follow as d is a metric and a supremum preserves these properties. The triangle inequality is also as immediate, but we go ahead and show it: $\rho(f, h) = \sup\{d(f(x), h(x))\} \leq \sup\{d(f(x), g(x)) + d(g(x), h(x))\} = \sup\{d(f(x), g(x))\} + \sup\{d(g(x), h(x))\} = \rho(f, g) + \rho(g, h)$. □

Lemma 1.43.15. $\overline{\rho}(f, g) = \min\{\rho(f, g), 1\}$*, mirroring our other bounded metric notation!*

Proof. If $\rho(f, g) > 1$, then there exists $x \in X$ such that $d(f(x), g(x)) > 1$, so $\overline{d}(f(x), g(x)) = 1$, and therefore $\overline{\rho}(f, g) = 1$.

If $\rho(f, g) \leq 1$, then $\overline{d}(f(x), g(x)) = d(f(x), g(x)) \leq 1$ for all x, so $\overline{\rho}(f, g) = \rho(f, g)$. □

Therefore, if $\mathcal{B}(X, Y)$ is complete under $\overline{\rho}$, then $\mathcal{B}(X, Y)$ is complete under ρ. Furthermore, if X is compact, then every continuous function $f : X \to Y$ is bounded, so $\mathcal{C}(X, Y) \subseteq \mathcal{B}(X, Y)$, and $\mathcal{C}(X, Y)$ is complete under ρ too.

Theorem 1.43.16. *If (X, d) is a metric space, then (X, d) embeds isometrically into a complete metric space; i.e., if $\Phi : (X, d_X) \to (Y, d_Y)$ is an embedding, then $d_Y(\Phi(a), \Phi(b)) = d_X(a, b)$.*

Proof. We see that $\mathcal{B}(X, \mathbf{R})$ is complete, as every bounded sequence of functions into $\mathbf{R}$ converges to a bounded function. We'll embed (X, d) into $\mathcal{B}(X, \mathbf{R})$.

Fix $x_0 \in X$. Then for all $a \in X$, define $\varphi_a(x) = d(x, a) - d(x, x_0)$. We'll use these φ_as to build an isometric embedding. First, we show that φ_a is bounded.

Since $d(x, a) \leq d(x, x_0) + d(x_0, a)$ by the triangle inequality, we see that $|\varphi_a(x)| = |d(x, a) - d(x, x_0)| \leq d(a, x_0)$, finite as d is a metric.

Now let's build the embedding. Let $\Phi : X \to \mathcal{B}(X, \mathbf{R})$ be $\Phi(a) = \varphi_a$. We claim that Φ is an isometric embedding into $(\mathcal{B}(X, \mathbf{R}), \rho)$. That is, we must show for all $a, b \in X$, $\rho(\varphi_a, \varphi_b) = d(a, b)$.

To see this, observe that

$$\begin{aligned}\rho(\varphi_a, \varphi_b) &= \sup_{x \in X}\{|\varphi_a(x) - \varphi_b(x)|\} \\ &= \sup_{x \in X}\{|d(x, a) - d(x, b)|\}\end{aligned}$$

So $\rho(\varphi_a, \varphi_b) \leq d(a, b)$. Now we just need to show the inequality in the other direction.

Let $x = a$, and we get $|\varphi_a(a) - \varphi_b(a)| = |d(a, a) - d(a, b)| = d(a, b)$. Since ρ is a supremum, it follows that $\rho(\varphi_a, \varphi_b) \geq d(a, b)$, and thus $\rho(\varphi_a, \varphi_b) = d(a, b)$, so Φ is an isometry. □

Definition 1.43.17. Let $X \subseteq Y$. If $f : X \to Y$ is an isometric embedding into a complete metric space, then $\overline{f(X)}$, the closure of the image of X, is called a **completion** of X.

1.44 Section 44: A Space-Filling Curve

Definitions:
Main Idea: Counter-intuitive at first, but clever.

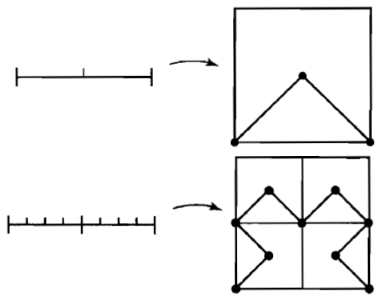
We know that $|I| = |I^2|$. We build a "space filling curve[6]" $f : I \to I^2$, where f is surjective and, moreover, continuous!

First, we give I^2 the square metric (**Definition 1.20.10**), $d(\overline{x}, \overline{y}) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$, and $\mathcal{C}(I, I^2)$ the uniform metric (**Definition 1.20.13**), $\rho(f, g) = \sup\{d(f(x), g(x)) \mid x \in I\}$. Note that $\mathcal{C}(I, I^2)$ is complete in ρ, since I^2 is closed, hence complete in d. So if we build a Cauchy sequence of continuous functions, then they will converge to a continuous function. We do this, and then show that the limit function is surjective.

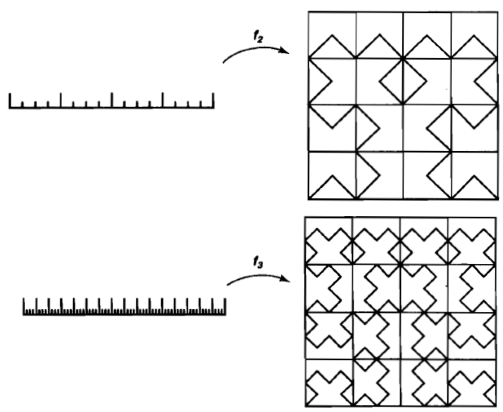
For f_0, take $[0, 1]$ and divide it into two pieces. Map the first piece linearly from the bottom left corner to the center of I^2, and the second linearly from the center to the bottom right corner. It is clear that f_0 is continuous.

For f_1, take I^2 and divide it into four equal parts. Take $[0, 1]$ and divide it into four pieces. Map the first piece into the bottom left corner, connecting its bottom left and top left corners, using the structure of f_0 described above. Then map the second piece of $[0, 1]$ into the top left corner of I^2, the third into the top right, and the fourth into the bottom right, connecting up all the ends so that f_1 is continuous. f_0 and f_1 are pictured below.

[6] This is the Peano curve!



Inductively, continue to subdivide I^2 and I and map in these triangluar regions. f_2 and f_3 are labeled below.



We get a sequence $(f_n) \in \mathcal{C}(I, I^2)$.

Lemma 1.44.1. *As we have constructed it, (f_n) is Cauchy.*

Proof. First, note that for a fixed x, $d(f_n(x), f_{n+1}(x)) \leq \frac{1}{2^n}$, because both points must lie in the same subsquare of length $\frac{1}{2^n}$. Therefore, for $m > n$, $d(f_n(x), f_m(x)) \leq \frac{1}{2^n} + \frac{1}{2^{n+1}} + ... + \frac{1}{2^{m-1}} \leq \frac{1}{2^{n-1}}$, which can be made arbitrarily small. Thus (f_n) is Cauchy. $\square$

Therefore, since $\mathcal{C}(I, I^2)$ is complete, $(f_n) \to f \in \mathcal{C}(I, I^2)$. All that remains to be seen is that f is surjective:

Lemma 1.44.2. *As we have constructed it, f is surjective.*

Proof. Let $\overline{x} \in I^2$; then we wish to show that $\overline{x} \in f(I)$. Fix an n; then for some $y \in I$, the distance from $f_n(\overline{y})$ to $\overline{x}$ is $\frac{1}{2^n}$, since they share some subsquare of width $\frac{1}{2^n}$. Let $\varepsilon > 0$ and let N be such that $\rho(f_N, f) < \frac{\varepsilon}{2}$ and $\frac{1}{2^N} < \frac{\varepsilon}{2}$. Then $d(\overline{x}, f(y)) \leq d(\overline{x}, f_N(y)) + d(f_N(y), f(y)) < \varepsilon$, so the ε-neighborhood of $\overline{x}$ intersects $f(I)$; i.e., $\overline{x} \in \overline{f(I)}$. But I is compact, so $f(I)$ is compact, and in a metric space, closed. So $\overline{f(I)} = f(I)$, and thus $\overline{x} \in f(I)$, as desired. $\square$

1.45 Section 45: Compactness in Metric Spaces

Definitions: totally bounded, equicontinuous, pointwise bounded
Main Idea: A lot of machinery in this section builds up to Arzela-Ascoli.

Definition 1.45.1. (X, d) is **totally bounded** if for all $\varepsilon > 0$ there exists a finite covering of X by ε-balls.

Note that totally bounded implies bounded. Not the converse, however:

Example 1.45.2. Consider $(\mathbf{R}, \overline{d})$. We see that it is bounded, but not totally bounded.

To see bounded, let $\varepsilon > 1$ and fix $x \in \mathbf{R}$. Then $\mathbf{R} \subseteq B_{\overline{d}}(x, \varepsilon)$.

To see not totally bounded, see that if $\varepsilon < 1$, then there does not exist a finite covering of $\mathbf{R}$ with ε-balls.

Theorem 1.45.3. *(X, d) is compact if and only if it is complete and totally bounded.*

Proof. If (X, d) is compact, then it is complete, because compact implies convergent subsequences, so **Lemma 1.43.3** has it. To see that it is totally bounded, let $\varepsilon > 0$ and cover X with ε-balls. Since X is compact, there is a finite subcover, and we have it.

Now assume (X, d) is complete and totally bounded. We show that X is sequentially compact (**Definition 1.28.4**), and **Theorem 1.28.5** gives us compactness.

Let $(x_n) \subseteq X$ be a sequence; we wish to show it has a convergent subsequence. Then cover X with 1-balls, finitely many by total boundedness. One of those balls B_1 must contain infinitely many points of (x_n). Pick any point in that ball; call it x_{j_1}. Then cover X by $\frac{1}{2}$-balls, and one ball B_2 has infinitely many points in $(x_n) \cap B_1$. Let x_{j_2} be a point in there, distinct from x_{j_1}. Inductively, we have B_k of radius $\frac{1}{2^k}$ and $x_{j_k} \in B_k$. Then $B_k \cap B_{k+1}$ has infinitely many points of (x_n). In particular, $d(x_{j_k}, x_{j_m}) \leq \frac{1}{2^{k-1}}$. Therefore, (x_{j_i}) is Cauchy, and as X is complete, (x_{j_i}) converges. We have constructed a convergent subsequence of an arbitrary sequence, so X is sequentially compact, therefore compact. $\square$

Definition 1.45.4. Let (Y, d) be a metric space. Let $\mathcal{F} \subseteq \mathcal{C}(X, Y)$. We say $\mathcal{F}$ is **equicontinuous** if for all $x_0 \in X$ and $\varepsilon > 0$, there exists an open neighborhood U of x_0 such that $f(U) \subseteq B_d(f(x_0), \varepsilon)$ for all $f \in \mathcal{F}$; i.e., for all $x \in U$, $d(f(x), f(x_0)) < \varepsilon$.

Definition 1.45.5. We say that $\mathcal{F} \subseteq \mathcal{C}(X, Y)$ is **pointwise bounded** if for all $x \in X$, $\{f(x) \mid f \in \mathcal{F}\} \subseteq Y$ is bounded.

Lemma 1.45.6. *If $\mathcal{F} \subseteq \mathcal{C}(X, Y)$ is totally bounded, then $\mathcal{F}$ is equicontinuous.*

Proof. Fix $x_0 \in X$, and ε such that $0 < \varepsilon < 1$. Let $\delta = \frac{\varepsilon}{3}$. Cover $\mathcal{F}$ by finitely many δ-balls in the uniform metric; i.e., $\mathcal{A} = \{B_{\overline{\rho}}(f_1, \delta), ..., B_{\overline{\rho}}(f_n, \delta)\} \supseteq \mathcal{F}$. Then for $x \in X$, there exists a neighborhood U of x such that $d(f_i(x), f_i(x_0)) < \delta$ for all $i \leq n$. Then given $f \in \mathcal{F}$, as $\mathcal{A}$ is a cover, we have $f \in B_{\overline{\rho}}(f_i, \delta)$ for some i, and so for $x \in U$,

$$\begin{aligned} \overline{d}(f(x), f_i(x)) &< \delta, \text{ as } \overline{\rho}(f, f_i) < \delta, \\ d(f_i(x_0), f_i(x)) &< \delta, \text{ by choice of } U, \text{ and} \\ \overline{d}(f_i(x_0), f(x_0)) &< \delta, \text{ as } \overline{\rho}(f, f_i) < \delta. \end{aligned}$$

Since $\delta < 1$, the metric d agrees with $\overline{d}$, so by the triangle inequality, we see that $d(f(x), f(x_0)) < \varepsilon$, so $f(U) \subseteq B_d(f(x_0), \varepsilon)$, and $\mathcal{F}$ is equicontinuous. $\square$

Lemma 1.45.7. *If X and Y are compact and $\mathcal{F} \subseteq \mathcal{C}(X, Y)$ is equicontinuous, then $\mathcal{F}$ is totally bounded.*

Proof. As X is compact, the sup-metric ρ (**Definition 1.43.13**) is defined, and ρ is totally bounded if and only if the uniform metric $\overline{\rho}$ (**Definition 1.20.13**) is totally bounded, so we show using ρ.

Fix $\varepsilon > 0$. Let $\delta = \frac{\varepsilon}{3}$. For all $a \in X$, there exists a neighborhood U_a of a such that $d(f(x), f(a)) < \delta$ for all $x \in U_a$ and $f \in \mathcal{F}$.

As X is compact, we can cover X by finitely many such $U_{a_1}, ..., U_{a_k}$. Furthermore, since Y is compact, we cover it with finitely many $V_1, ..., V_m$, where $\operatorname{diam}(V_i) < \delta$.

Let $J = \{\alpha \mid \{1, ..., k\} \to \{1, ..., m\}\}$. Then $|J| = m^k < \infty$, so for all $\alpha \in J$, if there exists $f \in \mathcal{F}$ such that $f(a_i) \in V_{\alpha(i)}$ for all $i = 1, ..., k$, then pick one such f and denote it f_α. Then let $J' \subseteq J$ be $\{\alpha \mid f_\alpha \in \mathcal{F}\}$. We now claim that $B_\rho(f_\alpha, \varepsilon)$ covers $\mathcal{F}$, and therefore $\mathcal{F}$ is totally bounded.

To see this, let $f \in \mathcal{F}$. There exists $\alpha \in J'$ such that $f(a_i) \in V_{\alpha(a_i)}$. We need to show that $\rho(f, f_\alpha) < \varepsilon$. Let $x \in X$, so $x \in U_{a_i}$ for some a_i. Then

$$\begin{aligned} d(f(x), f(a_i)) &< \delta, \text{ by choice of } U_{a_i} \text{ and equicontinuity of } \mathcal{F}, \\ d(f(a_i), f_\alpha(a_i)) &< \delta, \text{ as both lie in } V_{\alpha(i)} \text{ and } \operatorname{diam}(V_{\alpha(i)} < \delta, \text{ and} \\ d(f_\alpha(a_i), f_\alpha(x)) &< \delta, \text{ by choice of } U_{a_i} \text{ and equicontinuity of } \mathcal{F}. \end{aligned}$$

Therefore $d(f(x), f_\alpha(x)) < \varepsilon$, so $\rho(f, f_\alpha) < \varepsilon_\rho$, and $\mathcal{F}$ is totally bounded. $\square$

Theorem 1.45.8 (**Classical Arzela-Ascoli**). *Let X be compact. Let $(\boldsymbol{R}^n, d)$ be $\boldsymbol{R}^n$ with either the square metric or the Eucidean metric. Then $\mathcal{F} \subseteq \mathcal{C}(X, \boldsymbol{R}^n)$ has compact closure (i.e., $\overline{\mathcal{F}}$ is compact) if and only if $\mathcal{F}$ is equicontinuous and pointwise bounded.*

Proof. First, assume that $\mathcal{F}$ has compact closure. Let $\overline{\mathcal{F}} = \mathcal{G}$. Then $\mathcal{G}$ is compact. By **Theorem 1.45.3**, $\mathcal{G}$ is totally bounded, so $\mathcal{F}$ is. By **Lemma 1.45.6**, $\mathcal{F}$ is equicontinuous. Since $\mathcal{G}$ is bounded, $\mathcal{F}$ is bounded, so $\mathcal{F}$ is pointwise bounded.

Next, assume that $\mathcal{F}$ is equicontinuous and pointwise bounded. Our first goal is to show that this implies that $\overline{\mathcal{F}} = \mathcal{G}$ is equicontinuous and pointwise bounded.

To see that $\mathcal{G}$ is equicontinuous, fix $x_0 \in X$ and $\varepsilon > 0$. Then there exists a neighborhood U of x_0 such that $d(f(x_0, f(x)) < \frac{\varepsilon}{3}$ for all $x \in U$ and $f \in \mathcal{F}$. Let $g \in \mathcal{G}$. Then let f be such that $\rho(f, g) < \frac{\varepsilon}{3}$. Analogous to **Lemmas 1.45.6** and **1.45.7**, $d(g(x), g(x_0)) < \varepsilon$ for all $x \in U$. Therefore $\mathcal{G}$ is equicontinuous.

To see that $\mathcal{G}$ is pointwise bounded, fix $a \in X$. Call $\{f(a) \mid f \in \mathcal{F}\} = \mathcal{F}_a$. Then as $\mathcal{F}$ is pointwise bounded, there exists M such that $\operatorname{diam}(\mathcal{F}_a) \leq M$. Let $g, g' \in \mathcal{G}$. Then there exists $f, f' \in \mathcal{F}$ such that $\rho(f, g) < 1$ and $\rho(f', g') < 1$. Therefore, $d(g(a), g'(a)) \leq M + 2$, and $\mathcal{G}$ is pointwise bounded.

We now have that $\mathcal{G} = \overline{\mathcal{F}}$ is equicontinuous and pointwise bounded. We now claim that $\displaystyle\bigcup_{g \in \mathcal{G}} g(X) \subseteq [-N, N]^n$.

To see this, cover X by $U_{a_1}, ..., U_{a_k}$ where if $x \in U_{a_i}$ and $g \in \mathcal{G}$, then $d(g(x), g(a_i)) < 1$. Then call $\displaystyle\bigcup_{g \in \mathcal{G}} g(a_i) = \mathcal{G}_{a_i}$. $\mathcal{G}_{a_i}$ is bounded since $\mathcal{G}$ is pointwise bounded. Then consider $\displaystyle\bigcup_{i=1}^{k} \mathcal{G}_{a_i}$. This is bounded as the union of finitely many bounded sets is bounded. By construction, every $g(x)$ has distance at most 1 from some point in $\displaystyle\bigcup_{i=1}^{k} \mathcal{G}_{a_i}$. Therefore for some N, $\displaystyle\bigcup_{g \in \mathcal{G}} g(X) \subseteq [-N, N]^n$. Call $[-N, N]^n = Y$.

We now use everything developed thus far to show that $\mathcal{G}$ is compact. See that $\mathcal{G} \subseteq \mathcal{C}(X, Y)$. X is compact by hypothesis, and we have shown that Y is compact, so by **Lemma 1.45.7**, $\mathcal{G}$ is totally bounded. Since $\mathcal{G}$ is a closed subset of $\mathcal{C}(X, Y)$ which is complete, $\mathcal{G}$ is complete. Finally, by **Theorem 1.45.3**, $\mathcal{G}$ complete and totally bounded implies $\mathcal{G}$ is compact, as desired. $\square$

1.46 Section 46: Pointwise and Compact Convergence

Definitions: compact convergence topology, pointwise convergence topology, compact-open topology, evaluation map, homotopic

Main Idea: Perhaps these topics are best intuitively understood in a real analysis perspective. Get extra comfortable with them there, and then see how they generalize into abstract topological spaces.

We can define various topologies on $\mathcal{C}(X, Y) \subseteq Y^X$:

First, recall the uniform topology, induced by the uniform metric (**Definition 1.20.13**) on Y^X. Recall $\overline{\rho}(f, g) = \sup\{\overline{d}(f(x), g(x)) \mid x \in X\}$.

Definition 1.46.1. Let (Y, d) be a metric space and X a topological space. If $C \subseteq X$ is compact, $f \in Y^X$, and $\varepsilon > 0$, then a basis is given by $B_C(f, \varepsilon) = \{g \in Y^X \mid \sup\{d(f(x), g(x)) \mid x \in C\} < \varepsilon\}$. This basis generates the **compact**

convergence topology on Y^X. We sometimes say this is the topology of uniform convergence on compact sets (UCC).

Definition 1.46.2. Let $x \in X$ and let $U \subseteq Y$ be open. Let $S(x, U) = \{f \in Y^X \mid f(x) \in U\}$. Then a basis is given by $\left\{\bigcap_{i=1}^{n} S(x_i, U_i)\right\}$. This basis generates the **pointwise convergence topology on** Y^X.

We remark that $S(x, U) = {\pi_x}^{-1}(U)$, so the pointwise convergence topology is just the product topology on Y^X.

Proposition 1.46.3. *The pointwise convergence topology (product topology) is coarser than the compact convergence topology, which is coarser than the uniform topology. Furthermore, if X is compact, then by definition the compact convergence topology (UCC) is equivalent to the uniform topology.*

Theorem 1.46.4. *A sequence $(f_n) \to f$ in Y^X with the compact convergence topology if and only if $(f_n|_C) \to f|_C$ uniformly for all $C \subseteq X$ compact.*

Proof. First suppose $(f_n) \to f$ in UCC, and let $C \subseteq X$ be compact. Then let $\varepsilon > 0$. There exists N such that if $n \geq N$, then $f_n \in B_C(f, \varepsilon)$. Thus, $d(f_n(x), f(x)) < \varepsilon$ for $x \in C$ and $n \geq N$. Therefore $(f_n|_C) \to f|_C$ uniformly, as desired.

The converse is similar; for large n, $f_n \in B_C(f, \varepsilon)$. □

Theorem 1.46.5. *A sequence $(f_n) \to f$ in Y^X with the pointwise convergence topology if and only if $(f_n(x)) \to f(x)$ for all $x \in X$.*

Proof. Since the pointwise convergence topology is the product topology, this is exactly **Lemma 1.43.5**. □

Definition 1.46.6. Let X and Y be topological spaces. Let $C \subseteq X$ be compact and $U \subseteq Y$ be open. Then let $S(C, U) = \{f \in Y^X \mid f(C) \subseteq U\}$. Then a basis is given by $\{\bigcap_{i=1}^n S(C_i, U_i)\}$. This basis generates the **compact-open topology on** $\mathcal{C}(X, Y)$.

We remark that the pointwise convergence topology is coarser than the compact-open topology. Also note that this looks like pointwise convergence but doesn't require that (Y, d) is a metric space. In fact:

Theorem 1.46.7. *If (Y, d) is a metric space, then the compact convergence topology and the compact-open topology on $\mathcal{C}(X, Y)$ are the same.*

Proof. For a bit of notation, let $N_\varepsilon(A) = \bigcup_{a \in A} B_d(a, \varepsilon)$. We call $N_\varepsilon(A)$ the ε-neighborhood of A. For the proof, we will show that each basis is contained in the other.

To show that the compact-open topology is coarser than UCC, let $f \in S(C, U)$. Then as f is continuous, $f(C) \subseteq U$ is compact. Hence $f(C) \subseteq$

$N_\varepsilon(f(C)) \subseteq U$, since we can take $\varepsilon = \inf\{d(a, Y \setminus U) \mid a \in f(C)\}$. Therefore $f \in B_C(f,\varepsilon) \subseteq S(C,U)$, as desired.

To show that UCC is coarser than the compact-open topology, let $f \in B_C(f,\varepsilon)$. Then each $x \in X$ has a neighborhood V_x such that $f(\overline{V_x}) \subseteq U_x$ where $\operatorname{diam}(U_x) < \varepsilon$, since as f is continuous, $f(\overline{V_x}) \subseteq \overline{f(V_x)}$. As C is compact, we can cover C with finitely many $V_{x_1}, ..., V_{x_n}$, and set $C_i = C \cap \overline{V_{x_i}}$. C_i is compact, as it is a closed subspace of a compact space C. Therefore, $f \in S(C_1, U_{x_1}) \cap ... \cap S(C_n, U_{x_n}) \subseteq B_C(f,\varepsilon)$, as desired. □

Corollary 1.46.8. *The compact convergence topology on $\mathcal{C}(X,Y)$ doesn't depend on the metric on Y.*

Furthermore, if X is compact, then since the uniform topology agrees with the UCC topology on $\mathcal{C}(X,Y)$, the uniform topology doesn't depend on the metric on Y either.

Definition 1.46.9. We define the **evaluation map** $e : X \times \mathcal{C}(X,Y) \to Y$ by $e(x,f) = f(x)$.

Theorem 1.46.10. *Let X be locally compact and Hausdorff. Let $\mathcal{C}(X,Y)$ have the compact-open topology. Then the evaluation map is continuous.*

Recall locally compact in **Definition 1.29.2**.

Proof. Let V be a neighborhood of $e(x,f) = f(x)$ in Y. Since X is locally compact and Hausdorff, and f is continuous, by **Corollary 1.29.11** there exists a neighborhood $U \subseteq X$ of x such that $\overline{U}$ is compact and $f(\overline{U}) \subseteq V$. Therefore, $(x,f) \in U \times S(\overline{U}, V) \subseteq e^{-1}(V)$, and e is continuous. □

We make a quick remark here. If $f : X \times Z \to Y$ is a continuous function, then we get an induced function $\hat{f} : Z \to \mathcal{C}(X,Y)$ by $(\hat{f}(z))(x) = f(x,z) = f_z(x)$. We neglect to use this to define "induced functions," since we use that terminology a whole bunch in algebraic topology, and it doesn't quite coincide with this definition. For the definition we prefer to use, see **Definition 1.52.6**.

Theorem 1.46.11. *If $f : Z \to Y$ is continuous, then $\hat{f} : Z \to \mathcal{C}(X,Y)$ is continuous, using the compact-open topology. Furthermore, if X is locally compact and Hausdorff, then if $\hat{f}$ is continuous, then f is continuous.*

Proof. First, let $\hat{f}$ be continuous and X locally compact and Hausdorff. We show that f is continuous, because f is just the composition $f : X \times Z \xrightarrow{id_X \times \hat{f}} X \times \mathcal{C}(X,Y) \xrightarrow{e} Y$, i.e., $(x,z) \mapsto (x, f_z) \mapsto f_z(x) = f(x,z)$, so f is continuous.

Conversely, suppose f is continuous; we need to show that $\hat{f}$ is. Let $z_0 \in Z$, and $\hat{f}(z_0) \in S(C,U) \subseteq \mathcal{C}(X,Y)$. Then we seek a neighborhood W of z_0 such that $\hat{f}(W) \subseteq S(C,U)$; this will suffice to show that $\hat{f}$ is continuous.

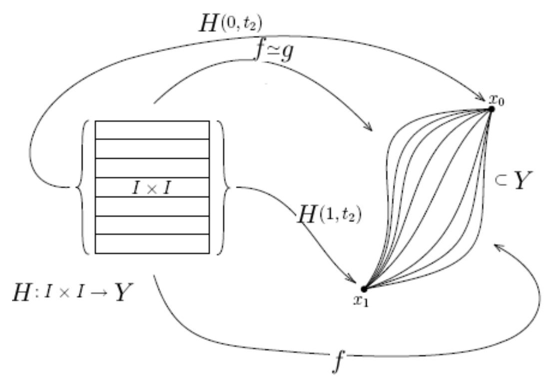
To that end, see that since $\hat{f}(z_0) \in S(C,U)$, we have $f_{z_0}(C) \subseteq U$; i.e., $f(C \times \{z_0\}) \subseteq U$. Then as f is continuous, $f^{-1}(U)$ is open, so $C \times \{z_0\} \subseteq f^{-1}(U) \cap (C \times Z) \subseteq C \times Z$. Then, by the **Tube Lemma 1.26.12**, there exists a neighborhood $W \subseteq Z$ of z_0 such that $C \times W \subseteq f^{-1}(U)$. We claim this W works.

Indeed, if $z \in W$, consider f_z. For all $x \in C$, we have $f_z(x) = f(x,z) \subseteq U$, as $(x,z) \in C \times W \subseteq f^{-1}(U)$. Thus $\hat{f}(W) \subseteq S(C,U)$, and $\hat{f}$ is continuous. $\square$

We turn to an application of induced maps: homotopy. We discuss homotopy more in **Section 51** and throughout algebraic topology, but we give a brief introduction here.

Definition 1.46.12. We say that $f, g : X \to Y$ are **homotopic**, and we write $f \simeq g$, if there exists $H : X \times [0,1] \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$. We call H a homotopy between f and g.

We see that H induces a continuous map $\hat{H} : [0,1] \to \mathcal{C}(X,Y)$ that continuously maps a time $t \in [0,1]$ to a continuous function from X to Y; i.e., $(\hat{H}(t))(x) = H_t(x) = H(x,t)$. A homotopy is thus a continuous, one-parameter family of functions. Below is a picture of a homotopy (actually a *path* homotopy; see **Definition 1.51.5**).



1.A Section A: Notes on Optimal Lipschitz Maps

Definitions: Lipschitz constant
Main Idea: We can use topology to talk about Lipschitz constants. This is actually unsurprising, since Ascoli's theorem is natural in a real analysis setting.

This section deals with an application of **Ascoli's Theorem 1.45.8**.

Definition 1.A.1. Suppose (X, d_X) and (Y, d_Y) are compact metric spaces. Given a continuous function $f \in \mathcal{C}(X,Y)$, we define the **Lipschitz constant of** f to be

$$L(f) = \sup\left\{\frac{d_Y(f(x), f(x'))}{d_X(x,x')} \mid x \neq x' \in X\right\} < \infty.$$

In other words, $d_Y(f(x), f(x')) \leq L(f) d_X(x,x')$.

Furthermore, given a closed subset $\mathcal{G} \subseteq \mathcal{C}(X,Y)$, we can consider $L(\mathcal{G}) = \inf\{L(f) \mid f \in \mathcal{G}\}$.

Example 1.A.2. Let X and Y be two spaces which are the same topologically, but differ in metric. Let $\mathcal{G} = \{h : X \to Y \mid h \text{ is a homeomorphism.}\}$. We naturally ask: is there a map $f \in \mathcal{G}$ such that $L(f) = L(\mathcal{G})$; i.e., is the infimum attained? Such a map f is called the optimal map.

To answer this question, consider the family $\mathcal{F} = \{f \in \mathcal{G} \mid L(f) \leq L(\mathcal{G})+1\}$. Since X and Y are compact metric spaces, we can use **Ascoli's Theorem 1.45.8** to see that if $\mathcal{F}$ is equicontinuous and pointwise bounded, then $\overline{\mathcal{F}}$ is compact. So we show $\mathcal{F}$ is equicontinuous and pointwise bounded.

For equicontinuous, let $x \in X$ and let $\varepsilon > 0$. Then if $x \in B_{d_X}\left(x_0, \frac{\varepsilon}{L(\mathcal{G}+1)}\right)$, then for all $f \in \mathcal{F}$, we have $d_Y(f(x), f(x_0)) \leq L(f)d_X(x, x_0) < (L(\mathcal{G}) + 1)\left(\frac{\varepsilon}{L(\mathcal{G})+1}\right) = \varepsilon$. Thus $\mathcal{F}$ is equicontinuous.

For pointwise bounded, since Y is compact, $\operatorname{diam}(Y) = D_Y < \infty$. Then if $\mathcal{F}_X = \{f(x) \mid f \in \mathcal{F}\}$, then $\operatorname{diam}(\mathcal{F}_X) \leq D_Y$, as $\mathcal{F}_X \subseteq Y$. Thus $\mathcal{F}$ is pointwise bounded.

Therefore **Ascoli's Theorem 1.45.8** gives that $\overline{\mathcal{F}}$ is compact. Now, let $f_n \in \mathcal{F}$ such that $L(f_n) \leq L(\mathcal{G}) + \frac{1}{n}$. Then $(f_n) \to f$ uniformly in $\overline{\mathcal{F}} \subseteq \mathcal{G}$. It just remains to be seen that $L(f) = L(\mathcal{G})$, but this is simple, involving e.g. the triangle inequality.

1.B Section B: The Cantor Set

Definitions: Cantor set, perfect, inverse limit system
Main Idea: The Cantor set is the only perfect, totally disconnected, compact metric space (up to homeomorphism).

There are several different characterizations of the Cantor set. Below, we give one.

Definition 1.B.1. The Cantor set is built inductively; let $C_0 = [0, 1]$. Then $C_1 = C_0 \backslash \left(\frac{1}{3}, \frac{2}{3}\right)$, removing the middle third. Then $C_2 = C_1 \backslash \left(\left(\frac{1}{9}, \frac{2}{9}\right) \cup \left(\frac{7}{9}, \frac{8}{9}\right)\right) = \frac{1}{3}C_1 \cup \left(\frac{2}{3} + \frac{1}{3}C_1\right)$, removing the middle thirds[7]. Continuing, we have

$$C_n = C_{n-1} \setminus \bigcup_{k=0}^{3^{n-1}} \left(\frac{1+3k}{3^n}, \frac{2+3k}{3^n}\right) = \frac{1}{3}C_{n-1} \cup \left(\frac{2}{3} + \frac{1}{3}C_{n-1}\right).$$

Then we define the **Cantor set** by $C = \bigcap_{n=0}^{\infty} C_n$.

Lemma 1.B.2. *The Cantor set is nonempty, closed, and compact.*

Proof. To see that the Cantor set is nonempty, note that the endpoints of every interval are contained in each C_n; e.g., 0 and $\frac{1}{3}$ are never removed. Thus $C \neq \emptyset$.

To see that the Cantor set is closed, note that it is the intersection of closed sets.

[7] Addition and multiplication on C_i is just defined by the operation applied to every $c_i \in C_i$.

To see that the Cantor set is compact, note that $C \subseteq [0,1]$, and C is closed and $[0,1]$ is compact. $\square$

Definition 1.B.3. We say that a set P is **perfect** if P is closed and every point is a limit point; i.e. $P = P'$.

Lemma 1.B.4. *The Cantor set C is totally disconnected and perfect.*

Recall **Definition 1.23.7**; totally disconnected means that the only connected components are singletons.

Proof. To see that C is totally disconnected, note that any connected subset of C is a connected subset of $[0,1]$, hence an interval. However, any nontrivial interval contains a subinterval of the form $\left(\frac{1+3k}{3^n}, \frac{2+3k}{3^n}\right)$ for some k and large enough n. But such an interval is removed during the construction of C, so we conclude there are no nontrivial connected subsets of C, and hence the only connected subsets are singletons.

To see that C is perfect, fix $x \in C$. We show that x is a limit point, which suffices. For all $n \in \mathbf{N}$, let I_n be the interval in C_n containing x. Then choose $x_n \in I_n$ by letting x_n be an endpoint of I_n, distinct from x. Then $d(x, x_n) \leq \frac{1}{3^n}$, so $x \in C'$. Thus C is perfect. $\square$

Definition 1.B.5. Given spaces $X_0, X_1, X_2, ...$ and maps $f_n : X_n \to X_{n-1}$, we have

$$... \xrightarrow{f_4} X_3 \xrightarrow{f_3} X_2 \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0.$$

Then we call $\{X_n, f_n\}$ a **inverse limit system**. We also define the inverse limit space

$$X_\infty = \varprojlim \{X_n, f_n\} \subseteq \prod_{n \geq 0} X_n,$$

which consists of (x_n) such that $f_n(x_n) = x_{n-1}$.

Example 1.B.6. At its most explicit,

$$\begin{aligned} X_\infty &= \{(..., x_4, x_3, x_2, x_1, x_0) \mid x_0 = f_1(x_1), x_1 = f_2(x_2), ..., x_{n-1} = f_n(x_n), ...\} \\ &\subseteq \prod_{n \geq 0} X_n. \end{aligned}$$

Example 1.B.7. Let $X_n = \mathbf{Z}/2^n\mathbf{Z} = \{0, 1, ..., 2^{n-1}\}$. Give X_n the discrete topology. Let $f_n : \mathbf{Z}/2^n\mathbf{Z} \to \mathbf{Z}/2^{n-1}\mathbf{Z}$ be $f_n(x) = x \mod 2^{n-1}$. Then X_∞ is

$$
\begin{tikzcd}
 & 0 \arrow[dr, "f_3"] & & & \\
 & 4 \arrow[r, "f_3"] & 0 \arrow[dr, "f_2"] & & \\
 & 2 \arrow[dr, "f_3"] & & 0 \arrow[ddr, "f_1"] & \\
 & 6 \arrow[r, "f_3"] & 2 \arrow[ur, "f_2"] & & \\
\cdots & & & & 0 \\
 & 1 \arrow[r, "f_3"] & 1 \arrow[dr, "f_2"] & & \\
 & 5 \arrow[ur, "f_3"] & & 1 \arrow[uur, "f_1"] & \\
 & 3 \arrow[r, "f_3"] & 3 \arrow[ur, "f_2"] & & \\
 & 7 \arrow[ur, "f_3"] & & &
\end{tikzcd}
$$

$$\cdots \xrightarrow{f_4} X_3 = \mathbf{Z}/8\mathbf{Z} \xrightarrow{f_3} X_2 = \mathbf{Z}/4\mathbf{Z} \xrightarrow{f_2} X_1 = \mathbf{Z}/2\mathbf{Z} \xrightarrow{f_1} X_0 = \mathbf{Z}/\mathbf{Z}$$

Then we see that a point in X_∞ is a path in the rooted binary tree; e.g., $(...5, 1, 1, 0) \in X_\infty$. We see that X_∞ is actually the 2-adics.

Lemma 1.B.8. *The following three spaces are homeomorphic.*

1. $\boldsymbol{Z}_{2^\infty} = \varprojlim \boldsymbol{Z}/2^n\boldsymbol{Z}$, *the* 2*-adics,*
2. $\prod_{\boldsymbol{N}} \{0, 1\}$ *where* $\{0, 1\}$ *has the discrete topology, and*
3. *the Cantor set* C*.*

Proof. Call $\prod\{0, 1\} = X$. Then we build a homeomorphism $f : C \to X$. Write $C_n = L_n \cup R_n$, where L_n is the left thirds of the remaining intervals and R_n is the right thirds of the the remaining intervals. In other words, If $C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$ and $C_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{3}{9}\right] \cup \left[\frac{6}{9}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$, then $L_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{6}{9}, \frac{7}{9}\right]$ and $R_2 = \left[\frac{2}{9}, \frac{3}{9}\right] \cup \left[\frac{8}{9}, 1\right]$.

Now define a sequence via functions $f_n : C_n \to \{0, 1\}$ by

$$f_n(x) = \begin{cases} 0 & \text{if } x \in L_n, \\ 1 & \text{if } x \in R_n. \end{cases}$$

Then let $f = (f_n)$, $f : C \to X$. We claim that f is a homeomorphism. To see this, first note that f is a bijection; any point in the Cantor set is an infinite choice of lefts and rights, and any point in X is an infinite choice of 0s and 1s. Next, we see that f is continuous. Let $U = \prod U_i \subseteq X$, a basis member for the product topology on X. Then for all (finitely many) i such that $U_i \neq \{0, 1\}$, let

$$Z_i = \begin{cases} R_i & \text{if } U_i = \{1\}, \\ L_i & \text{if } U_i = \{0\}. \end{cases}$$

Then $f^{-1}(U) = C \cap \left(\bigcap_{i=1}^{n} Z_n \right)$. So f is continuous.

As f is a continuous bijection between compact Hausdorff spaces, by **Theorem 1.26.11**, f is a homeomorphism.

Now let's build another homeomorphism $g : X \to \mathbf{Z}_{2^\infty}$. Let $g((x_n)) = (m_n)$ where $m_0 = 0$ and

$$m_n = \begin{cases} m_{n-1} & \text{if } x_n = 0; \text{i.e., the top branch of the 2-adics,} \\ m_{n-1} + 2^{n-1} & \text{if } x_n = 1; \text{i.e., the bottom branch of the 2-adics.} \end{cases}$$

Then as before, g is bijective, since a choice of 0 or 1 in X is exactly a choice of top or bottom in $\mathbf{Z}_{2^\infty}$, and it can be seen that g is continuous. By **Theorem 1.26.11**, we again have a homeomorphism. $\square$

In other words, we remarked before we defined the Cantor set that there are other characterizations. Here are two. Our goal for this section is to completely describe the Cantor set in terms of its properties, namely a metric space that is perfect, totally disconnected, and compact. We first develop some lemmas:

Lemma 1.B.9. *Let X be a metric space that is totally disconnected and compact. Then for all $a, b \in X$ such that $a \neq b$, there exist nonempty disjoint open sets A and B in X such that $a \in A$, $b \in B$, and $X = A \cup B$. We say a and b are separated in X.*

Note that this is stronger than disconnectedness, since we can separate any two points. Just based on the terminology, we should expect this to be true.

Proof. Suppose not. Then let $\{H_\alpha\}$ be the collection of closed subsets of X in which a and b are not separated. See that $\{H_\alpha\} \neq \emptyset$, since a and b are not separated in X.

Then $\{H_\alpha\}$ is partially ordered by inclusion, so by the **Minimal Principle**, there exists a minimal simply ordered chain $\{K_\alpha\} \subseteq \{H_\alpha\}$. Now consider $K = \bigcap_\alpha K_\alpha$, which is closed as each K_α is. We claim that $K \in \{H_\alpha\}$; i.e., we claim a and b are not separated in K.

> To prove this claim, suppose not. Then let A and B be the nonempty disjoint open subsets such that $a \in A$, $b \in B$, and $K = A \cup B$. Then A and B are closed in K, since $K \setminus A = B$ and vice versa. Since

K is closed, A and B are closed in X as well. Since X is a metric space, X is normal, so we can separate closed sets. To that end, fix disjoint open sets $U, V \subseteq X$ such that $A \subseteq U$ and $B \subseteq V$. Then $K_\alpha \neq (K_\alpha \cap U) \cup (K_\alpha \cap V)$, as a and b cannot be separated in K_α. Thus, let $L_\alpha = K_\alpha \cap (X \setminus (U \cup V)) \neq \emptyset$. We see that $\{L_\alpha\}$ has the finite intersection property, as the L_αs are simply ordered. So since X is compact, $\bigcap L_\alpha \neq \emptyset$. But this implies $K \cap (X \setminus (U \cup V)) \neq \emptyset$, contradicting the fact that $K \subseteq U \cup V$. Therefore, a and b are not separated in K.

Then, since X is totally disconnected, K is not connected, so $K = C_1 \cup C_2$ where C_1 and C_2 are nonempty disjoint open subsets. As we cannot separate a and b in K, without loss of generality we let $a, b \in C_1$.

However, $C_1 \in \{H_\alpha\}$, since if $C_1 = A \cup B$ with $a \in A$ and $b \in B$, then $K = A \cup (B \cup C_2)$ where $a \in A$ and $b \in B \cup C_2$, so a and b are separated. But this is a contradiction, because we supposed K was minimal with respect to separating a and b. This contradiction implies that all a and b are separated in X. □

Lemma 1.B.10. *Let X be a metric space that is totally disconnected and compact. Then for all $a \in X$ and $U \subseteq X$ a neighborhood of a, there exists a clopen set $V \subseteq X$ such that $a \in V \subseteq U$.*

Proof. Consider $X \setminus U$, which is compact, as X is. For all $b \in X \setminus U$, there exists A_b and B_b in X such that A_b and B_b are disjoint and clopen, $a \in A_b$ and $b \in B_b$, and $X = A_b \cup B_b$, by **Lemma 1.B.9**. Since $X \setminus U$ is compact, $X \setminus U \subseteq B_{b_1} \cup ... \cup B_{b_n}$, which is a finite subcover of the cover $\{B_b \mid b \in X \setminus U\}$, since X is compact. Then let $V = \bigcap_{i=1}^{n} A_{b_i}$. We claim this works. See that V is clopen, since it is a finite intersection of clopen sets. See also that $a \in V$, since $a \in A_{b_i}$ for all i. And see that $V \subseteq U$, since $X \setminus U$ is covered by B_{b_i}s and A_{b_i}s are disjoint. □

Lemma 1.B.11. *Let X be a metric space that is perfect, totally disconnected, and compact*[8]. *Then for all $U \subseteq X$ open and nonempty and for all $n \in \boldsymbol{N}$, there exist disjoint nonempty open sets $U_1, ... U_n$ such that $U = U_1 \cup ... \cup U_n$.*

Proof. We proceed via induction. For the base case where $n = 1$, the proof is super obvious; let $U_1 = U$.

For the inductive step, suppose $U = U_1 \cup ... \cup U_n$, and each U_i is open and nonempty and disjoint from all others, i.e., $U_i \cap U_j = \emptyset$ if $i \neq j$. Now, as X is perfect, singleton sets are not open. Since X is totally disconnected, U_n is not

[8] I cannot figure out if compactness is necessary or not. It isn't explicitly used, but perhaps that U can be finitely decomposed requires it. My gut, however, says that it is not necessary, but I leave it in the hypotheses just in case. This section is isolated from all others, so we won't need to refer to this lemma again except in the proof of the characterization of the Cantor set, which we know is compact, so the extra hypothesis does no harm.

connected. Therefore $U_n = V \cup W$ for nonempty disjoint open subsets V and W, and hence $U = U_1 \cup ... \cup U_{n-1} \cup V \cup W$, as claimed. $\square$

Lemma 1.B.12. *Let $\{X_n, f_n\}$ be an inverse limit system where each X_n is a compact Hausdorff space. Then $X_\infty = \varprojlim\{X_n, f_n\}$ is a compact Hausdorff space.*

Proof. Since each X_n is Hausdorff, $\prod X_n$ is Hausdorff, and since $X_\infty \subseteq \prod X_n$, X_∞ is Hausdorff. This is just **Theorem 1.31.7**. Compactness remains.

By **Tychonoff's Theorem 1.37.3**, $\prod X_n$ is compact. It thus suffices to show that X_∞ is closed, so we show its complement is open. Let $x \in \prod X_n \setminus X_\infty$. Then there exist $p, q \in \mathbf{N}$ such that $p \geq q$ and $f_{pq}(x_p) \neq x_q$. Let $U, V \subseteq X_n$ be disjoint and open neighborhoods such that $f_{pq}(x_p) \in U$ and $x_q \in V$. Then choose $U' \subseteq X_p$ to be an open neighborhood of x_p such that $f_{pq}(U') \subseteq U$. Now let

$$Y_n = \begin{cases} U' & \text{if } n = p, \\ V & \text{if } n = q, \\ X_n & \text{otherwise.} \end{cases}$$

Finally, let $W = \prod Y_n \subseteq \prod X_n$. Then W is open in the product topology and is a neighborhood of $x \in \prod X_n \setminus X_\infty$. Thus X_∞ is closed, and hence compact, as desired. $\square$

We are now ready to characterize the Cantor set. The proof is involved, so we first explain the structure of the proof in steps, and then prove each step.

Theorem 1.B.13. *Any two perfect, totally disconnected, compact metric spaces are homeomorphic. As we know C is a perfect, totally disconnected, compact metric space, any such space is the Cantor set.*

Proof. Let X be such a space. We will prove the theorem in four steps.

Step One: We show that for all $n \in \mathbf{N}$, there exists a finite covering $\mathcal{O}_n$ of X by disjoint clopen subsets such that $\operatorname{diam}(\mathcal{O}_n) < \frac{1}{n}$ and $\mathcal{O}_{n+1}$ is a refinement of $\mathcal{O}_n$; i.e., for all sets $U \in \mathcal{O}_{n+1}$, there exists $V \in \mathcal{O}_n$ such that $U \subseteq V$.

Step Two: We build an inverse limit system $\{X_n, f_n\}$.

Step Three: We show that X is homeomorphic to $\varprojlim\{X_n, f_n\}$.

Step Four: We show that if X and Y are perfect, totally disconnected, compact metric spaces, then $\varprojlim\{X_n, f_n\}$ is homeomorphic to $\varprojlim\{Y_n, g_n\}$.

Step One: We construct the $\mathcal{O}_n$ inductively. We start with $\mathcal{O}_1$. For all $x \in X$, there exists clopen sets V_x such that $x \in V_x \subseteq B_d\left(x, \frac{1}{2}\right)$, by **Lemma 1.B.10**. Then let $V_1, ..., V_m$ be a finite subcover of $\{V_x\}$. Then let $U_1 = V_1$, $U_2 = V_2 \setminus V_1$, $U_3 = V_3 \setminus (V_1 \cup V_2)$, and so on, so that $U_j = V_j \setminus \bigcup_{i=1}^{j-1} V_i$. Then set $\mathcal{O}_1 = \{U_j\}$.

We can continue in this manner. For all $x \in X$, there exists clopen V_x such that $x \in V_x \subseteq B_d\left(x, \frac{1}{4}\right) \cap U_i$ for i uniquely depending on x- uniquely, since all

U_i are disjoint. Then we can take a finite subcover of $\{V_x\}$ and do the same trick, creating $\{U_n\}$ which become $\mathcal{O}_2$.

For $\mathcal{O}_n$, see that for all $x \in X$, there exists clopen V_x such that $x \in V_x \subseteq B_d\left(x, \frac{1}{2^n}\right) \cap U_i$ for $U_i \in \mathcal{O}_{n-1}$. By construction, we have exactly what we needed; $\mathcal{O}_n$ is a finite covering of X by disjoint clopen sets with $\text{diam}(\mathcal{O}_n) < \frac{1}{n}$, and by construction $\mathcal{O}_{n+1}$ is a refinement of $\mathcal{O}_n$.

Step Two: We build our inverse limit system. For each $n \in \mathbf{N}$, let X_n be the space whose points are the subsets in $\mathcal{O}_n$. Give X_n the discrete topology. Then define $f_n : X_n \to X_{n-1}$ by $f_n(U) = V$ where $V \in \mathcal{O}_{n-1}$ is such that $U \subseteq V$. The function f is well-defined, since sets in $\mathcal{O}_{n-1}$ are disjoint, and a subsequent $\mathcal{O}_n$ is a refinement of $\mathcal{O}_{n-1}$. With $\{X_n, f_n\}$ constructed, now consider $X_\infty = \varprojlim\{X_n, f_n\} \subseteq \prod_{n\in\mathbf{N}} X_n$.

Step Three: We claim that X is homeomorphic to X_∞. To see this, let $U_i \in \mathcal{O}_i$ and let $U_i \subseteq U_{i-1}$. Let $x = (..., U_3, U_2, U_1) \in X_\infty$. We define a function $h : X_\infty \to X$ by $h(x) = \bigcap_{i=1}^{\infty} U_i$.

This function h is well-defined, as $\{U_i\}_{i=1}^{\infty}$ have the finite intersection property, so $\bigcap_{i=1}^{\infty} U_i \neq \emptyset$, and as $\text{diam}(U_i) < \frac{1}{i}$, the intersection contains at most one point, so it contains exactly one point. We now need to see that h is a homeomorphism. We show that it is injective, surjective, and continuous.

For injective, let $x, y \in X_\infty$ such that $x \neq y$. Then some coordinate of x, call it U, is disjoint from coordinate V of y. Since $h(x) \in U$ and $h(y) \in V$, $h(x) \neq h(y)$. So h is injective.

For surjective, let $x \in X$. Then for all n, since $\mathcal{O}_n$ is a cover of X, there exists some $U_n \in \mathcal{O}_n$ such that $x \in U_n$. Then $x = \bigcap_{n=1}^{\infty} U_n = h((..., U_3, U_2, U_1))$. So h is surjective.

For continuous, note that $\bigcup_{n\in\mathbf{N}} \mathcal{O}_n$ is a basis for the topology on X. Then given $U \in \mathcal{O}_n$, $h^{-1}(U)$ consists of sequences with U as the nth coordinate. Since $\{U\} \subseteq X_n$ is open, $h^{-1}(U) \subseteq X_\infty$ is open. So h is continuous.

Since X is compact Hausdorff, X_n is for all n, and by **Lemma 1.B.12**, X_∞ is compact Hausdorff. By **Theorem 1.26.11**, since h is a continuous bijection between compact Hausdorff spaces, h is a homeomorphism, as desired.

Step Four: We show any two totally disconnected, perfect, compact metric spaces are homeomorphic. Let X and Y be two such spaces, and let $(\mathcal{O}_n)$ and $(\mathcal{P}_n)$ be sequences of finite clopen disjoint covers where every element has diameter less than $\frac{1}{n}$. In other words, write $\mathcal{O}_1 = \{U_{1,1}, U_{1,2}, ..., U_{1,k}\}$ and $\mathcal{P}_1 = \{V_{1,1}, V_{1,2}, ..., V_{1,l}\}$.

We inductively alter these sequences to make them comparable. If $k = l$, then set ${\mathcal{O}_1}' = \mathcal{O}_1$ and ${\mathcal{P}_1}' = \mathcal{P}_1$. If, however, without loss of generality we have $k > l$, we use **Lemma 1.B.11** to decompose $V_{1,1}$ into $k - l + 1$ disjoint open sets ${V_{1,1}}', {V_{1,2}}', ..., {V_{1,k-l+1}}'$. Then let ${\mathcal{P}_1}' = \{{V_{1,1}}', ..., {V_{1,k-l+1}}', V_{1,2}, ..., V_{1,l}\}$, and let ${\mathcal{O}_1}' = \mathcal{O}_1$. Then $|{\mathcal{O}_1}'| = |{\mathcal{P}_1}'|$.

We'd like to keep going in this manner, but note that by decomposing $V_{1,1}$, we've messed up the refinement property. However, we claim that there exists $m_{\mathcal{O}} \in \mathbf{N}$ such that no set of diameter less than $\frac{1}{m_{\mathcal{O}}}$ intersects two sets in ${\mathcal{O}_1}'$.

> To see this claim, suppose not; then there exist $(x_m) \subseteq U_1$ and $(y_m) \subseteq U_2$ such that $(d(x_m, y_m))_m \to 0$. Then, by compactness, we have $(x_m) \to x$ and $(y_m) \to x$ where $x \in \overline{U_1} \cap \overline{U_2} = U_1 \cap U_2$, since U_i is already closed. But note that if $i \neq j$, then U_i is disjoint from U_j, so $U_1 \cap U_2 = \emptyset$. This contradiction implies the claim.

Similarly, there exists $m_{\mathcal{P}} \in \mathbf{N}$ such that no set of diameter less than $\frac{1}{m_{\mathcal{P}}}$ intersects two sets in ${\mathcal{P}_1}'$. There, we can set $m = \max\{m_{\mathcal{O}}, m_{\mathcal{P}}\}$. Therefore, $\mathcal{O}_m$ refines ${\mathcal{O}_1}'$, and $\mathcal{P}_m$ refines ${\mathcal{P}_1}'$. All that we need to show, then, is an inductive-esque argument that we can decompose an $\mathcal{O}_*$ and $\mathcal{P}_*$ to be comparable, and then see that a refinement exists for some later coverings. Then we'll have two new sequences $({\mathcal{O}_n}')$ and $({\mathcal{P}_n}')$ that retain the properties of $(\mathcal{O}_n)$ and $(\mathcal{P}_n)$ and are comparable for all n.

So fix $i \in \mathbf{N}$. Consider $U_{1,i} \in {\mathcal{O}_1}'$ and $V_{1,i} \in {\mathcal{P}_1}'$. Let $r_i = \{U \in \mathcal{O}_m mid U \subseteq U_{1,i}\}$ and let $s_i = \{V \in \mathcal{P}_m \mid V \subseteq V_{1,i}\}$. Then let $a = \max\{|r_i|, |s_i|\}$. Then assume without loss of generality that $|r_i| \geq |s_i|$. Then, as before, decompose some set in s_i into $a - |s_i + 1$ subsets. Do this for each i, and call the resulting covers ${\mathcal{O}_2}'$ and ${\mathcal{P}_2}'$. Notice that we have $|{\mathcal{O}_2}'| = |{\mathcal{P}_2}'|$.

Continue in this manner. Then $({\mathcal{O}_n}')$ and $({\mathcal{P}_n}')$ have the same combinatorics. Therefore, we have bijections $\varphi_n : {X_n}' \to {Y_n}'$ such that the following diagram commutes:

$$\begin{array}{ccc} {X_n}' & \xrightarrow{f_n} & {X_{n-1}}' \\ \big\downarrow{\scriptstyle \varphi_n} & & \big\downarrow{\scriptstyle \varphi_{n-1}} \\ {Y_n}' & \xrightarrow{g_n} & {Y_{n-1}}' \end{array}$$

This implies there exists an induced continuous function $\varphi : X_\infty \to Y_\infty$ such that $\varphi((x_n)) = (\varphi_n(x_n))$. From this we can conclude that $\varprojlim\{X_n, f_n\}$ is homeomorphic to $\varprojlim\{Y_n, g_n\}$, and we are finally done. $\square$

1.51 Section 51: Homotopy of Paths

Definitions: simply connected, nulhomotopic, path product
Main Idea: We develop path homotopy as an equivalence relation on paths. We define an operation on equivalence classes of paths which is associative, has identity, and has inverses. Where could we possibly be going?...

The motivating question from here on out is to develop topological invariants which describe when two spaces are (not) homeomorphic. We begin with an example:

Example 1.51.1. Is $\mathbf{R}^n$ homeomorphic to $\mathbf{R}^n \setminus \{\overline{0}\}$?

For $n = 0$, obviously not, since a point is not homeomorphic to the empty space.

For $n = 1$, the answer is again no, because $\mathbf{R}$ is path connected, but $\mathbf{R} \setminus \{0\}$ is not.

For $n = 2$, the answer is currently not so obvious. Pretty much every property we've described so far that holds for $\mathbf{R}^2$ also holds for $\mathbf{R}^2 \setminus \{\overline{0}\}$; connectedness, path connectedness, etc. However, we'll build something that does differentiate $\mathbf{R}^2$ from $\mathbf{R}^2 \setminus \{\overline{0}\}$.

We remark now that there is another way to describe path connectedness: a space X is path connected if every continuous map $f : S^0 \to X = f : \{-1, 1\} \to X$ extends to a map $F : D^1 \to X = F : [-1, 1] \to X$.

Definition 1.51.2. We say that a space X is **simply connected** if it is path connected and every continuous map $f : S^1 \to X$ extends to a map $F : D^2 \to X$.[9]

We remark that being simply connected is a topological invariant, because we can just compose with homeomorphisms. We will see that $\mathbf{R}^2$ is simply connected, while $\mathbf{R}^2 \setminus \{\overline{0}\}$ is not.

Example 1.51.3. While we haven't yet built up the machinery, we'll see that $\mathbf{R}^2 \setminus \{x_1\}$ is not simply connected. What about $\mathbf{R}^2 \setminus \{x_1, x_2\}$? It too will not be simply connected, but we would like that it be different from $\mathbf{R}^2 \setminus \{x_1\}$. So already this new invariant has reached a roadblock. However, we'll introduce the fundamental group (**Definition 1.52.1**) to remedy this.

• • •

So we've given a brief overview of where we intend to go. Now let's build up the theory. First recall **Definition 1.46.12**, homotopic maps; $f : X \to Y \simeq g : X \to Y$ if there exists a continuous homotopy $H : X \times [0, 1] \to Y$ such that at time 0, $H(x, 0) = f(x)$, and at time 1, $H(x, 1) = g(x)$.

Definition 1.51.4. If f is homotopic to a constant function, we say that f is **nulhomotopic**.

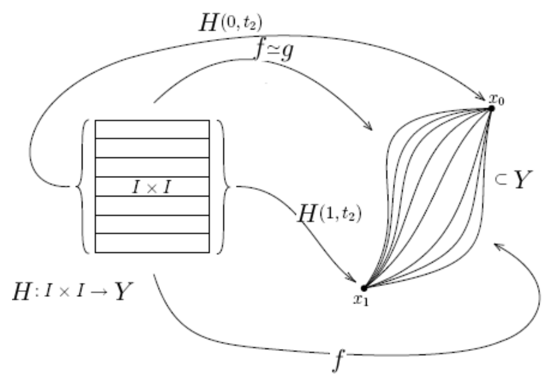
There is a special type of homotopicity we have already alluded to:

Definition 1.51.5. Let $f, g : I \to X$ be a path with initial point $x_0 = f(0)$ and ending at a terminal point $x_1 = f(1)$. We say that f and g are **path homotopic**, **homotopic as paths**, **homotopic relative to endpoints**, or

[9]Note that we could continue to generalize, and we will, when we discuss higher homotopy groups!

homotopic rel endpoints if there exists a continuous homotopy $H : I^2 \to X$ such that for all $s \in [0,1]$, $H(s,0) = f(s)$ and $H(s,1) = g(s)$, and for all $t \in [0,1]$, $H(0,t) = x_0$ and $H(1,t) = x_1$. In other words, the endpoints are fixed for all functions in the homotopy, and we homotope from f to g. We write $f \simeq_p g$, but often just write $f \simeq g$ when it is clear we mean path homotopic, which we usually do.

The picture given before was of a path homotopy; we give it again here.



Lemma 1.51.6. *The relations $\simeq$ and $\simeq_p$ are equivalence relations.*

Proof. We show it for $\simeq$; the other is nearly identical.

Reflexive is clear; to see $f \simeq f$, take the homotopy which keeps f constant for all time.

Symmetric is also clear; if $f \simeq g$ via a homotopy $H(x,t)$, then $H(x, 1-t)$ shows $g \simeq f$.

Transitive is also also clear; if $f \simeq g$ via $H(x,t)$ and $g \simeq h$ via $H'(x,t)$, then the homotopy

$$\mathbf{H}(x,t) = \begin{cases} H(x,2t) & \text{if } 0 \le t \le \frac{1}{2}, \\ H'(x,2t-1) & \text{if } \frac{1}{2} \le t \le 1, \end{cases}$$

shows $f \simeq h$. $\square$

Thus we typically only work with equivalence classes of paths, $[f]$. We can define a product on equivalence classes:

Definition 1.51.7. If $f, g : I \to X$ where $f(1) = x_1 = g(0)$, then we can define the **product** $f \cdot g = h : I \to X$ by

$$h(t) = \begin{cases} f(2t) & \text{if } 0 \le t \le \frac{1}{2}, \\ g(2t-1) & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

Note that h is continuous, by the **Pasting Lemma 1.18.9**.

Lemma 1.51.8. *This induces a well-defined operation on path-homotopy classes:*

$$[f] \cdot [g] = [f \cdot g].$$

Proof. Let $f_0 \simeq f_1$ via F and $g_0 \simeq g_1$ via G. Then we get a homotopy $H : I \times I \to X$, defined by

$$\mathbf{H}(x,t) = \begin{cases} F(2x,t) & \text{if } 0 \leq t \leq \frac{1}{2}, \\ G(2x-1,t) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Note that this is another way to compose homotopies (versus **Lemma 1.51.6**). Here, F and G are glued horizontally. Above, they were glued one atop the other vertically.

At $x = \frac{1}{2}$, $F(1,t) = x_1$ and $G(0,t) = x_1$, so by the **Pasting Lemma 1.18.9**, H is continuous. □

Theorem 1.51.9. *The product $[f] \cdot [g]$ has the following properties:*

1. *It is associative; i.e., $([f] \cdot [g]) \cdot [h] simeq [f] \cdot ([g] \cdot [h])$.*
2. *Identities exist (up to endpoint). Given $x \in X$, define $e_x : I \to X$ by $e_x(s) = x$. Then $[f] \cdot [e_{x_1}] \simeq [f] \simeq [e_{x_0}] \cdot [f]$.*
3. *Inverses exist. Given $f : I \to X$, define $\overline{f} : I \to X$ by $\overline{f}(s) = f(1-s)$. Then $[f] \cdot [\overline{f}] \simeq [e_{x_0}]$ and $[\overline{f}] \cdot [f] \simeq [e_{x_1}]$.*

Proof. The key tool is to reparametrize the homotopy, as we have done in the previous proofs. Let $\varphi : I \to I$ with $\varphi(0) = 0$ and $\varphi(1) = 1$. Then $f\varphi \simeq f$, via $F(s,t) = f((1-t)\varphi(s) + ts)$. So see that:

1. $([f] \cdot [g]) \cdot [h] = [f] \cdot ([g] \cdot [h])$, since we can reparametrize the speed: a homotopy that does $[f] \cdot [g]$ on time $t \in \left[0, \frac{1}{2}\right]$ and $[h]$ on time $t \in \left[\frac{1}{2}, 1\right]$ is the same as a homotopy that does $[f]$ on time $t \in \left[0, \frac{1}{2}\right]$ and does $[g] \cdot [h]$ on time $t \in \left[\frac{1}{2}, 1\right]$.
2. A homotopy that does $[f]$ on time $t \in [0,1]$ is the same as a homotopy that does $[f]$ on time $t \in \left[0, \frac{1}{2}\right]$ and does $[e_{x_1}]$ on time $t \in \left[\frac{1}{2}, 1\right]$, and is the same as a homotopy that does $[e_{x_0}]$ on time $t \in \left[0, \frac{1}{2}\right]$ and does $[f]$ on time $t \in \left[\frac{1}{2}, 1\right]$.
3. Let $i : I \to I$ be
$$i(s) = \begin{cases} 2s & \text{if } 0 \leq t \leq \frac{1}{2}, \\ 1-2s & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$
Then $i \simeq e_0$ via $H(s,t) = ti(s)$. Then see that $[f] \cdot [\overline{f}] = fi \simeq fe_0 = e_{x_0}$.

□

1.52 Section 52: The Fundamental Group

Definitions: fundamental group, induced map
Main Idea: The fundamental group is a powerful topological invariant.

Definition 1.52.1. The **fundamental group of** X **based at** $x_0 \in X$ is $\pi_1(X, x_0) = \{[f] \mid f : I \to X, f(0) = x_0 = f(1)\}$ with operation $\cdot$, the product on path classes.

By **Lemma 1.51.8** and **Theorem 1.51.9**, this is well-defined.

Example 1.52.2. $\pi_1(\mathbf{R}^n, \overline{0})$ is the trivial group, which we write as $\{1\}$, $\{0\}$, or 0, depending on how the wind blows.

To see this, given any path $f : I \to \mathbf{R}^n$ with $f(0) = \overline{0} = f(1)$, consider the homotopy $F(s,t) = (1-t)f(s)$. This is a straight-line homotopy down to the based point $\overline{0}$; we see that $F(s, 0) = f(s)$ and $F(s, 1) = e_{\overline{0}}(s) = \overline{0}$. Thus any $f \simeq e_{\overline{0}}$, hence $[f] = 1$. In fact, this same argument holds for any convex region of $\mathbf{R}^n$.

We remark that since $f(0) = x_0 = f(1)$, f factors through S^1:

$$\begin{tikzcd}
(I, \partial I) \arrow[rr] \arrow[dr] & & (X, x_0) \\
& (S^1, s_0) \arrow[ur] &
\end{tikzcd}$$

In other words, $\pi_1(X, x_0)$ can be thought of as the group of equivalence classes of loops based at $s_0 \in X$.

A natural question to ask is: what is the dependence on x_0? To answer this question, consider $\alpha : I \to X$ from x_0 to x_1. Then the map α induces a map $\hat{\alpha}$ defined by $\hat{\alpha}([f]) = [\overline{\alpha}][f][\alpha]$. This is well-defined since concatenation is well-defined when it's defined. Essentially, $\hat{\alpha}$ starts at x_1, goes to x_0 using $\overline{\alpha}$, preforms the loop f, then returns to x_1 via α. In other words, $\hat{\alpha}$ has turned a loop based at x_0, $[f]$, into a loop based at x_1, $\hat{\alpha}([f])$.

Theorem 1.52.3. $\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1)$ *is an isomorphism between groups.*

Proof. We show $\hat{\alpha}$ is a bijective homomorphism. To see homomorphism, let $[f], [g] \in \pi_1(X, x_0)$. Then $\hat{\alpha}([f])\hat{\alpha}([g]) = [\overline{\alpha}][f][\alpha][\overline{\alpha}][g][\alpha] = [\overline{\alpha}][f][g][\alpha] = \hat{\alpha}([f][g])$. To see bijection, we show the inverse map. It is $\hat{\beta}$ induced by $\beta = \overline{\alpha}$, hence $\overline{\beta} = \alpha$. $\square$

Corollary 1.52.4. *If* X *is path connected, then* $\pi_1(X, x_0) \cong \pi_1(X, x_1)$ *for all* $x_0, x_1 \in X$*. If not, then the fundamental group at* x_0 *is isomorphic to the the fundamental group at* x_1 *where* x_1 *is in the same path component as* x_0*.*

As a result, if X is path connected, we can (and often will) be lazy about specifying base point. We often omit it entirely, once we become more comfortable with fundamental groups in later sections.

We caution here that there is no canonical isomorphism between fundamental groups. Suppose both $\alpha, \beta : I \to X$ are paths from x_0 to x_1. Then $[\overline{\alpha}\beta] \in$

$\pi_1(X, x_1)$. So

$$\begin{aligned}\hat{\alpha}([f]) &= [\overline{\alpha}][f][\alpha] \\ &= [\overline{\alpha}][\beta\overline{\beta}][f][\beta\overline{\beta}][\alpha] \\ &= [\overline{\alpha}\beta][\overline{\beta}][f][\beta][\overline{\alpha}\beta]^{-1} \\ &= [\overline{\alpha}\beta]\hat{\beta}([f])[\overline{\alpha}\beta]^{-1}.\end{aligned}$$

So $\hat{\alpha} = C_{\overline{\alpha}\beta}\hat{\beta}$, where $C_{\overline{\alpha}\beta}$ is conjugation by $[\overline{\alpha}\beta]$. Note that this shows that if the fundamental group is abelian, then conjugation is trivial, so $\hat{\alpha} = \hat{\beta}$, and we do have a canonical isomorphism.

We can restate **Definition 1.51.2** of a space X being simply connected. We now say that X is simply connected if it is path connected and $\pi_1(X, x_0) = \{1\}$ for any $x_0 \in X$ (since X is path connected).

Lemma 1.52.5. *If X is simply connected and $\alpha, \beta : I \to X$ have the same endpoints x_0 and x_1, then $\alpha \simeq \beta$.*

Proof. Since $\overline{\alpha}\beta$ is a loop based at x_1, $\overline{\alpha}\beta \simeq e_{x_1}$, i.e., $\overline{\alpha}\beta$ is nulhomotopic as X is simply connected. Thus $\alpha \simeq \alpha e_{x_1} \simeq \alpha\overline{\alpha}\beta \simeq e_{x_0}\beta \simeq \beta$, as desired. □

Definition 1.52.6. If $h : (X, x_0) \to (Y, y_0)$ is a continuous function, we define an **induced map** $h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ by $h_*([f]) = [hf]$. In other words, we have

$$\begin{tikzcd}(I, \partial I) \arrow[r, "f"] \arrow[rr, bend right, "hf"'] & (X, x_0) \arrow[r, "h"] & (Y, y_0).\end{tikzcd}$$

This is well-defined, since if $f \simeq g$ via a homotopy F, then $hf \simeq hg$ via hF. Also, note that h is a homomorphism, as $(hf)(hf) = h(fg)$.

Note the following properties of induced maps:

1. $(id_X)_* = id_{\pi_1(X,x_0)}$.

2. $(kh)_* = k_*h_*$ whenever $(X, x_0) \xrightarrow{h} (Y, y_0) \xrightarrow{k} (Z, z_0)$.

In category theory terms, this shows that π_1 is a covariant functor from the category of pointed topological spaces to the category of groups.[10]

Corollary 1.52.7. *If $h : (X, x_0) \to (Y, y_0)$ is a homeomorphism, then h_* is an isomorphism.*

So homeomorphic topological spaces have the same fundamental group. This is nice to know, because otherwise the fundamental group is essentially useless for our stated goals in specific, and algebraic topology in general.

[10] I don't have the category theory knowledge to know what this means explicitly, but essentially it just means continuous in topology with based point is equivalent to homomorphism in group.

Proof. Let $k : (Y, y_0) \to (X, x_0)$ be the inverse to h; i.e., $kh = id_X$ and $hk = id_Y$. By the above two properties, $k_* h_* = id_{\pi_1(X,x_0)}$ and $h_* k_* = id_{\pi_1(Y,y_0)}$. Thus h_* and k_* are inverse homomorphisms, hence isomorphisms. In other words, h_* is an isomorphism with inverse k_*. $\square$

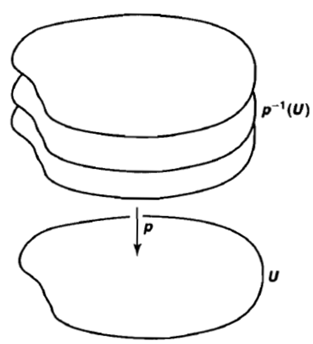
Warning: the converse is definitely not true! $\mathbf{R}$ and $\mathbf{R}^2$ have isomorphic fundamental groups, both 0, but they are certainly not homeomorphic.

1.53 Section 53: Covering Spaces

Definitions: covering map
Main Idea: We define covering spaces and see several examples. In coming sections we will see why we care.

Definition 1.53.1. A continuous function $p : \widetilde{X} \to X$ is a **covering map** if for all $x \in X$ there is a neighborhood of x $U \subseteq X$ such that $p^{-1}(U)$ is a disjoint union of sets $\coprod V_\alpha$ where $p|_{V_\alpha} : V_\alpha \to U$ is a homeomorphism. In other words, $p^{-1}(U)$ is homeomorphic to $U \times S$ where S is discrete.

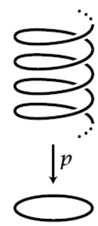
The picture is that of a stack of records or pancakes.



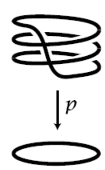
Note that $\widetilde{X}$ and p together describe the covering space. In fact, $(\widetilde{X}, X, p, S)$ is a fiber bundle (**Definition 2.14.24**) with discrete fiber. See **Example 2.14.26**.

Example 1.53.2. Trivially, let $\widetilde{X} = X \times \{1, 2, ..., n\}$ and let $p : \widetilde{X} \to X$ be $p(x, i) = x$. In other words, $\widetilde{X}$ is the disjoint union of n copies of X.

Example 1.53.3. Less trivially, the map $p : \mathbf{R} \to S^1$ defined by $p(x) = (\cos(2\pi x), \sin(2\pi x)) = e^{2\pi i x}$ is a covering map. To see this, see that if $U \subseteq S^1$ is $U = \{(1, \theta) \mid -\frac{\pi}{2} < \theta < \frac{\pi}{2}\}$, then $p^{-1}(U) = \bigcup_{n \in \mathbf{Z}} \left(n - \dfrac{1}{4}, n + \dfrac{1}{4} \right) = \bigcup_{n \in \mathbf{Z}} V_n$ for disjoint V_n. Then $p|_{V_n} : \left(n - \frac{1}{4}, n + \frac{1}{4}\right) \to U$ is a homeomorphism. Repeat for any rotations of U.



Example 1.53.4. Not only are there other coverings of S^1, there are other connected coverings of S^1. Consider $p : S^1 \to S^1$, where $p(z) = z^n$ with $n \neq 0$. Below is pictured one.



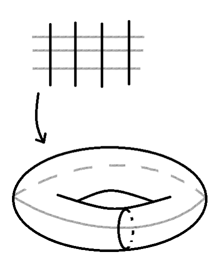
Theorem 1.53.5. *If $p : \widetilde{X} \to X$ is a covering map, $X_0 \subseteq X$, and $\widetilde{X_0} = p^{-1}(X_0))$, then $p_0 = p|_{\widetilde{X_0}} : \widetilde{X_0} \to X_0$ is a covering map.*

Proof. Let $x_0 \in X_0$ and let $U \subseteq X$ be a neighborhood of x_0 such that $p^{-1}(U) = \coprod V_\alpha$ where V_α is homeomorphic to U. Then $p_0{}^{-1}(U \cap X_0) = \coprod V_\alpha \cap \widetilde{X_0}$, and each $V_\alpha \cap \widetilde{X_0}$ is homeomorphic to $U \cap X_0$. $\square$

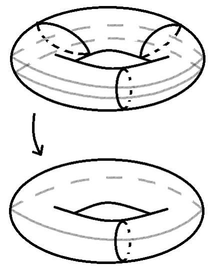
Theorem 1.53.6. *If $p_1 : \widetilde{X} \to X$ and $p_2 : \widetilde{Y} \to Y$ are covering maps, then so is $(p_1 \times p_2) : \widetilde{X} \times \widetilde{Y} \to X \times Y$.*

Proof. If $p_1{}^{-1}(U) = \displaystyle\coprod_\alpha V_\alpha$ and $p_2{}^{-1}(U') = \displaystyle\coprod_\beta V_\beta{}'$, then $(p_1 \times p_2)^{-1}(U \times U') = \coprod_{\alpha,\beta} V_\alpha \times V_\beta{}'$. $\square$

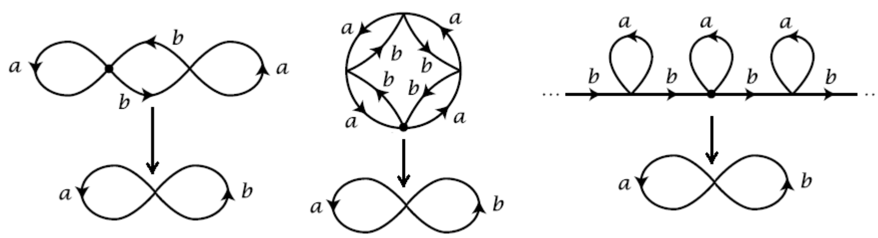
Example 1.53.7. $p : \mathbf{R}^2 \to S^1 \times S^1$ where $(x, y) \mapsto (e^{2\pi i x}, e^{2\pi i y}) \in \mathbf{C}^2$ is a covering map.



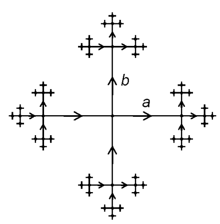
Example 1.53.8. $p : S^1 \times S^1 \to S^1 \times S^1$ where $(z, w) \mapsto (z^2, w^3)$ is a covering map.



Example 1.53.9. Let's see some covers of the figure eight graph. Note that the figure eight graph is a subspace of the torus; it is the black line and gray line in **Example 1.53.7** (it is also $S^1 \vee S^1$; see **Definition 2.2.7**). This means that the covers in **Example 1.53.7** and **Example 1.53.8** also cover the figure eight, by **Theorem 1.53.5**. We can see other covers though:



We can also get a simply connected cover, the Cayley graph on two generators:



1.54 Section 54: The Fundamental Group of the Circle

Definitions: lift, lifting correspondence
Main Idea: We show that it is $\mathbf{Z}$ by building the machinery of lifts.

Definition 1.54.1. Let $p : \widetilde{X} \to X$ be continuous. A **lift** of a continuous function $f : Y \to X$ is a continuous map $\widetilde{f} : Y \to \widetilde{X}$ such that $f = p\widetilde{f}$. In other words, a lift is the map $\widetilde{f}$ such that the following diagram commutes:

$$\begin{tikzcd} & \widetilde{X} \arrow[d, "p"] \\ Y \arrow[ur, dotted, bend left, "\widetilde{f}"] \arrow[r, "f"] & X \end{tikzcd}$$

Theorem 1.54.2 (**Path Lifting Property**). *Given a covering space $p : \widetilde{X} \to X$, a path $f : I \to X$, and a point $\widetilde{x_0} \in p^{-1}(f(0)) = p^{-1}(x_0)$, there exists a unique lift $\widetilde{f} : I \to \widetilde{X}$ such that $\widetilde{f}(0) = \widetilde{x_0}$.*

Proof. We explicitly construct such a lift $\widetilde{f}$.

Cover X with open sets $\{U\}$ such that $p^{-1}(U) = \coprod V_\alpha$. Subdivide $[0, 1]$ into $0 = s_0 < s_1 < ... < s_n = 1$ such that $f([s_i, s_{i+1}])$ is contained in some U. This is possible by the **Lebesgue Covering Lemma 1.27.5**. Our goal is to construct $\widetilde{f}$ in pieces without choice based on these intervals $[s_i, s_{i+1}]$.

Define $\widetilde{f}(0) = \widetilde{x_0}$, and further suppose that $\widetilde{f}$ is defined for $0 \leq s \leq s_i$. Then $f([s_i, s_{i+1}]) \subseteq U$ for some set U. Let $p^{-1}(U) = \coprod V_\alpha$, and then $p(\widetilde{f}(s_i)) = f(s_i) \in U$. Hence $\widetilde{f}(s_i) \in p^{-1}(U)$, so $\widetilde{f}(s_i) \in V_{\alpha_0}$ for some α_0. Then extend $\widetilde{f}$ over $[s_i, s_{i+1}]$ by $\widetilde{f}(s) = \left(p|_{V_{\alpha_0}}\right)^{-1}(f(s))$.

Use the homeomorphism of the covering space to extend the path locally; then, since I is compact and we make progress on defining $\widetilde{f}$ at each step, we see that $\widetilde{f} : I \to \widetilde{X}$ exists.

Now we see that $\widetilde{f}$ is unique, and we are done. The idea here is that we had no choice in constructing $\widetilde{f}$. So suppose $\widetilde{\widetilde{f}}$ is another lift of f such that $\widetilde{\widetilde{f}}(0) = \widetilde{x_0} = \widetilde{f}(0)$. Then suppose $\widetilde{\widetilde{f}}(s) = \widetilde{f}(s)$ for $0 \leq s \leq s_i$. Then $\widetilde{\widetilde{f}}(s_i) = \widetilde{f}(s_i) \in V_{\alpha_0}$. As $\widetilde{\widetilde{f}}([s_i, s_{i+1}]) \subseteq p^{-1}(U)$ and $[s_i, s_{i+1}]$ is closed, then $\widetilde{\widetilde{f}}([s_i, s_{i+1}]) \subseteq V_{\alpha_0}$. Hence for all $s \in [s_i, s_{i+1}]$, $\widetilde{\widetilde{f}}(s) \in p^{-1}(f(s)) \cap V_{\alpha_0} = \{\widetilde{f}(s)\}$. Thus the lift is unique. □

Proposition 1.54.3 (**Homotopy Lifting Property**). *Given a covering space $p : \widetilde{X} \to X$, a homotopy $H : Y \times I \to X$, and a lift of $f_0 = H(-, 0)$, $\widetilde{f_0} : Y \to \widetilde{X}$, there exists a unique lift $\widetilde{H} : Y \times I \to \widetilde{X}$.*

We remark that the **Path Lifting Property 1.54.2** is a special case of the **Homotopy Lifting Property 1.54.3**, where Y is a point. However, if we were to prove the **Homotopy Lifting Property**, we use the **Path Lifting Property**.

Corollary 1.54.4. *Let $p : \widetilde{X} \to X$ be a covering space. If $f, g : I \to X$ are path homotopic and $\widetilde{f}, \widetilde{g} : I \to \widetilde{X}$ are lifts with $\widetilde{f}(0) = \widetilde{g}(0)$, then $\widetilde{f} \simeq \widetilde{g}$ and in particular, $\widetilde{f}(1) = \widetilde{g}(1)$.*

Proof. Let $F : I \times I \to X$ be a path homotopy of f. Lift it to $\widetilde{F} : I \times I \to \widetilde{X}$ where $\widetilde{F}(0, t) = \widetilde{f}(0) = \widetilde{g}(0)$. Let's confirm that $\widetilde{F}$ is a path homotopy of $\widetilde{f}$ and $\widetilde{g}$.

Since $p(\widetilde{F}(s, 0)) = F(s, 0) = f(s)$, $\widetilde{F}(-, 0)$ is a lift of f starting at $\widetilde{f}(0)$. Thus $\widetilde{F}(s, 0) = \widetilde{f}(s)$ by the uniqueness of lifting paths. Likewise, $\widetilde{F}(-, 1) = \widetilde{g}$. And finally, as $p(\widetilde{F}(1, t)) = F(1, t) = f(1)$, we see that $\widetilde{F}(1, t) \in p^{-1}(f(1))$, which is a discrete space. Hence $\widetilde{F}(1, t)$ is constant and $\widetilde{F}(1, t) = \widetilde{f}(1) = \widetilde{g}(1)$. □

Definition 1.54.5. Fix $\widetilde{x_0} \in p^{-1}(x_0)$. We get a map $\varphi : \pi_1(X, x_0) \to p^{-1}(x_0)$ where $\varphi([f]) = \widetilde{f}(1)$, where $\widetilde{f} : I \to \widetilde{X}$ is the lift of f with $\widetilde{f}(0) = \widetilde{x_0}$. We call φ the **lifting correspondence**. By **Corollary 1.54.4**, this is well-defined.

Example 1.54.6. If $p : \mathbf{R} \to S^1$, $p(x) = e^{2\pi i x}$, is a covering space and f is a loop based at $1 \in S^1$, then $f : I \to S^1$ takes $\partial I \mapsto 1$. Then $\widetilde{f} : I \to \mathbf{R}$ takes $\partial I \mapsto m, n \in \mathbf{Z}$. In fact, $\widetilde{f}(0) = 0$, and $\varphi(f) = \widetilde{f}(1) \in \mathbf{Z}$.

For a general case, the path-lifting property lets us consider a function $\varphi : \{\text{loops based at } x_0\} \to p^{-1}(x_0)$. Choose some $\widetilde{x_0} \in p^{-1}(x_0)$. Then $\varphi(f) = \widetilde{x_1}$. In diagrams:

$$\begin{tikzcd}
\widetilde{X} \arrow[d, "p"] & \widetilde{x_0} \arrow[r, bend left, "\widetilde{f}"] & \widetilde{x_1} \\
X & x_0 \arrow[loop right, "f"] &
\end{tikzcd}$$

Theorem 1.54.7. *If $\widetilde{X}$ is simply connected, then φ is a bijection.*

Proof. Since $\widetilde{X}$ is path connected, for all $\widetilde{x_1} \in {p_1}^{-1}(x_0)$, there exists $f : I \to \widetilde{X}$ such that $f(0) = \widetilde{x_0}$ and $f(1) = \widetilde{x_1}$. Then $[pf] \in \pi_1(X, x_0)$ and $\varphi([pf]) = \widetilde{x_1}$. Thus φ is surjective.

For injective, see that if $\varphi([f]) = \varphi([g])$, then $\widetilde{f} \simeq \widetilde{g}$ as $\widetilde{X}$ is simply connected. Then composing with p gives us $f \simeq g$, so φ is injective. Thus φ is a bijection. □

Theorem 1.54.8. $\pi_1(S^1, 1) \cong \mathbf{Z}$.

Proof. Since $p : \mathbf{R} \to S^1$ with $p(x) = e^{2\pi i x}$, $\mathbf{R}$ is simply connected, and $p^{-1}(1) = \mathbf{Z}$, we have a bijection $\varphi : \pi_1(S^1, 1) \to \mathbf{Z}$. Choose $0 \in p^{-1}(1)$.

We show that φ is a homomorphism, and thus an isomorphism. Let $[f], [g] \in \pi_1(S^1, 1)$ and let $\widetilde{f}, \widetilde{g} : I \to \mathbf{R}$ start at 0. Then $n = \widetilde{f}(1) = \varphi([f])$ and $m = \widetilde{g}(1) = \varphi([g])$. Consider $\widetilde{\widetilde{g}}(s) = n + \widetilde{g}(s)$, which is also a lift of g, starting at n. Thus $\widetilde{f} \cdot \widetilde{\widetilde{g}}$ is a lift of $f \cdot g$ starting at 0. Then $\varphi([fg]) = \widetilde{f} \cdot \widetilde{\widetilde{g}}(1) = \widetilde{\widetilde{g}}(1) = n + \widetilde{g}(1) = n + m = \varphi([f]) + \varphi([g])$. Therefore, φ is a homomorphism, hence isomorphism, and $\pi_1(S^1, 1) \cong \mathbf{Z}$.

Note that $\langle 1 \rangle$ is a generator for $\mathbf{Z}$, and $w_1 : I \to S^1$ where $w_1(x) = e^{2\pi i x}$ is a generator for $\pi_1(S^1, 1)$. □

1.55 Section 55: Retractions and Fixed Points

Definitions: retraction
Main Idea: Retractions are important to know. Brouwer's Fixed Point Theorem is nice, and we generalize it further later.

Definition 1.55.1. A **retraction** is a map $r : X \to A$ where $A \subseteq X$ such that $r|_A = id_A$. In other words, if $\iota : A \to X$ is the inclusion map, then $r\iota = id_A$. The retraction is a left-inverse to ι.

If a retraction exists, we say that A is a retract of X.

Example 1.55.2. $D^n = \{\overline{x} \in \mathbf{R}^n \mid \|\overline{x}\| \le 1\}$ is a retract of $\mathbf{R}^n$. See the retraction

$$r(\overline{x}) = \begin{cases} \frac{\overline{x}}{\|\overline{x}\|} & \text{if } \|\overline{x}\| \ge 1, \\ \overline{x} & \text{if } \|\overline{x}\| \le 1. \end{cases}$$

r is continuous by the **Pasting Lemma 1.18.9**, and $r|_{D^n} = id_{D^n}$.

Example 1.55.3. S^{n-1} is a retract of $\mathbf{R}^n \setminus \{\overline{0}\}$. The retraction is $r(\overline{x}) = \frac{\overline{x}}{\|\overline{x}\|}$.

Lemma 1.55.4. *If A is a retract of X, i.e., $r\iota = id_A$, then the homomorphisms induced by inclusion, $\iota_* : \pi_1(A, a_0) \to \pi_1(X, x_0)$ and $r_* : \pi_1(X, x_0) \to \pi_1(A, a_0)$, are injective and surjective, respectively.*

Proof. We show that ι_* is injective; that r_* is surjective is pretty much the same.

Since $r\iota = id_A$, we have $r_*\iota_* = (r\iota)_* = (id_A)_* = id_{\pi_1(A,a_0)}$. Since $id_{\pi_1(A,a_0)}$ is injective, ι_* is too. $\square$

Theorem 1.55.5. S^1 *is not a retract of* D^2.

Proof. Suppose it is. Then the inclusion induced homomorphism $\pi_1(S^1, 1) \xrightarrow{\iota_*} \pi_1(D^2, 1)$ is injective. However, $\pi_1(S^1, 1) \cong \mathbf{Z}$, while $\pi_1(D^2, 1) \cong 0$. There does not exist an injective homomorphism from $\mathbf{Z}$ to 0. $\square$

Theorem 1.55.6 (Brouwer's Fixed Point Theorem). *If $f : D^2 \to D^2$ is continuous, then there exists a fixed point of f; i.e., there exists $\overline{x} \in D^2$ such that $f(\overline{x}) = \overline{x}$.*

Proof. Suppose not. Then the ray from $f(\overline{x})$ to $\overline{x}$ intersects S^1 in some point $R(\overline{x})$. We need to show that R is continuous.

To see this, note that the ray is parametrized by $l(\overline{x}, t) = \overline{x}t + (1-t)f(\overline{x})$. Then consider $\overline{x} = (x_1, x_2)$ and $f = (f_1, f_2)$. Then l hits the circle S^1 when $(tx_1 + (1-t)f_1(\overline{x}))^2 + (tx_2 + (1-t)f_2(\overline{x}))^2 = 1$. This is a quadratic expression in terms of t. So let $Q(\overline{x})$ be the positive root of the polynomial. Then $Q : D^2 \to \mathbf{R}$ varies continuously by varying coefficients. Then $R(\overline{x}) = l(\overline{x}, Q(\overline{x}))$ is a retraction $R : D^2 \to S^1$. However, this contradicts **Theorem 1.55.5**. Thus there does exist a fixed point of f. $\square$

We note that this does hold for lower dimensions; the **Intermediate Value Theorem 1.23.11** is just a guarantee that S^0 is not a retract of D^1. We will see later that this holds for higher dimensions as well; see **Theorem 2.8.41**.

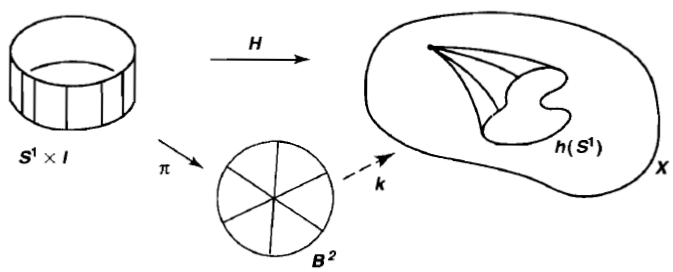
Lemma 1.55.7. *Let $h : S^1 \to X$ be continuous. The following are equivalent:*

1. *h is nulhomotopic; i.e., if $c : S^1 \to X$ is a constant map, then $h \simeq c$,*

2. *h extends to a map $g : D^2 \to X$; i.e., $h = g\iota$ where $\iota : S^1 \to D^2$ is the inclusion map, and*

3. *h_* is the trivial homomorphism.*

Proof. To see that 1. implies 2., we let $H : S^1 \times I \to X$ be a homotopy between h and a constant map c. Then h factors through a quotient map $k\pi$ collapsing $S^1 \times \{1\}$ to $\overline{0} \in D^2$ and then homotoping. See the following picture.



To see 2. implies 3., note that if $h = g\iota$, then $h_* = g_*\iota_*$. Since $\pi_1(D^2, 1) = 0$, we have that $g_* : \pi_1(D^2, 1) \to \pi_1(X, g(1))$ is trivial, so therefore h_* is trivial as well.

Finally, for 3. implies 1. Let $w_1(x) = e^{2\pi i x}$, the generator for $\pi_1(S^1, 1)$. Then $hw_1 : I \to X$ is nulhomotopic as paths, since $[hw_1] = h_*[w_1] = 1$.

Let $H : I \times I \to X$ be the nulhomotopy. Since $H(0, t) = H(1, t)$ for all t, the homotopy factors through the quotient

$$\begin{tikzcd}
I \times I \arrow[rr, "H"] \arrow[dr, "\text{quotient}"'] & & X \\
& I \times I \big/ (0,t) \sim (1,t) = S^1 \times I \arrow[ur, "\overline{H}"'] &
\end{tikzcd}$$

This is a nulhomotopy between h and the constant map, so h is nulhomotopic. ∎

1.56 Section 56: The Fundamental Theorem of Algebra

Definitions:
Main Idea: An important theorem, and interestingly, more topological than algebraic.But there's a million proofs anyways.

Theorem 1.56.1 (**The Fundamental Theorem of Algebra**)**.** *A polynomial equation $p(x) = x^n + a_{n-1}x^{n-1} + ... + a_1x + a_0 = 0$, where $n \geq 1$ and $a_i \in \boldsymbol{C}$ has a solution in $\boldsymbol{C}$.*

Proof. We break the proof into four steps.

In the first step, consider the map $f : S^1 \to S^1$ defined by $f(z) = z^n$, $n \in \mathbf{Z}$. Then f insudes a homomorphism $f_* : \pi_1(S^1, 1) \to \pi_1(S^1, 1)$, i.e., $f : \mathbf{Z} \to \mathbf{Z}$. We claim that f_* is multiplication by n. In lieu of an explicit proof, just see that f wraps the circle clockwise around itself n times, and f_* is uniquely defined by taking $1 \mapsto n$, the generator of each fundamental group.

In the second step, we see that the map $g : S^1 \to \mathbf{C} \setminus \{\overline{0}\}$ by $g(z) = z^n$ is not nulhomotopic as long as $n \neq 0$. To do this, let $r : \mathbf{C} \setminus \{\overline{0}\}$ be a retraction. Then $f = rg$, so $f_* = r_* g_*$. By step one, f_* is not trivial, so by **Lemma 1.55.7**, g_* is not trivial. Thus g is not nulhomotopic.

In the third step, we prove the Fundamental Theorem of Algebra in a specific case. Assume that $|a_{n-1}| + |a_{n-2}| + ... + |a_1| + |a_0| < 1$. We produce a solution in $D^2 \subseteq \mathbf{C}$. Suppose no solution exists in D^2. Then p defines a continuous function $p : D^2 \to \mathbf{C} \setminus \{\overline{0}\}$. Let $h = p|_{S^1} : S^1 \to \mathbf{C} \setminus \{\overline{0}\}$. As h is a retraction from the disk, it extends to the disk, so h is nulhomotopic. Let's show that $h \simeq g$, which invokes a contradiction, since g is not nulhomotopic and h is. The homotopy is $H : S^1 \times I \to \mathbf{C} \setminus \{\overline{0}\}$ given by $H(z, t) = z^n + t(a_{n-1} z^{n-1} + ... + a_1 z + a_0)$. Then $|H(z,t)| \geq |z^n| - |t(a_{n-1} z^{n-1} + ... + a_1 z + a_0)| \geq 1 - t(|a_{n-1}| + ... + |a_1| + |a_0|) > 0$. Thus $h \simeq g$, a contradiction.

In the fourth step, we finally prove the general case and the theorem. Let $x = cy$ for $c > 0$. Then $(cy)^n + a_{n-1}(cy)^{n-1} + ... + a_1 cy + a_0 = 0$. Equivalently, $y^n + \frac{a_{n-1}}{c} y^{n-1} + ... + \frac{a_1}{c^{n-1}} y + \frac{a_0}{c^y} = 0$. Then for large enough c, this has a solution $y_0 \in D^2$, by step three. Thus for $x_0 = cy_0$, x_0 is a solution to the original polynomial. $\square$

1.57 Section 57: The Borsuk-Ulam Theorem

Definitions:

Main Idea: If f maps each point of earth to two continuous values, then there are antipodal points with both values equal.

Theorem 1.57.1 (The Borsuk-Ulam Theorem). *If $f : S^2 \to \boldsymbol{R}^2$, then there exists $\overline{x} \in S^2$ such that $f(\overline{x}) = f(-\overline{x})$. Equivalently, there must exist antipodal points with the same value.*

Two antipodal points on earth have the same temperature and pressure.

Proof. Suppose not. Then $g(\overline{x}) = \frac{f(\overline{x}) - f(-\overline{x})}{\|f(\overline{x}) - f(-\overline{x})\|}$ defines a continuous function $g : S^2 \to S^1$. Notice that g is an odd map; i.e., $g(-\overline{x}) = -g(\overline{x})$. Now consider $p : I \to S^2$ given by $p(s) = (\cos(2\pi s), \sin(2\pi s))$. This path is nulhomotopic, as it is a retraction to the equator, and can be extended ot the upper hemisphere, a disk. Now let $h = gp : I \to S^1$. Notice that h is nulhomotopic as p is. Then we have that for $0 \leq s \leq \frac{1}{2}$, $h\left(s + \frac{1}{2}\right) = g(-p(s)) = -g(p(s)) = -h(s)$.

Then lift h to $\widetilde{h} : I \to \mathbf{R}$ where $\widetilde{h}(0) = 0$. Then since h is nulhomoptic, $\widetilde{h}(1) = 0$. However, $\widetilde{h}\left(s + \frac{1}{2}\right) = \widetilde{h}(s) + \frac{m}{2}$ for some odd integer m. Since $\widetilde{h}$

depends continuously on s and m takes discrete values, m must be constant. Therefore, $\widetilde{h}(1) = \widetilde{h}\left(\frac{1}{2}\right) + \frac{m}{2} = \widetilde{h}(0) + \frac{m}{2}\frac{m}{2} = m \neq 0$, not an odd integer[11]. Thus by contradiction, Borsuk-Ulam is proven. □

Corollary 1.57.2. *Given two bounded polygonal regions in $\boldsymbol{R}^2$, there exists a single line that bisects both into equal area regions.*

Note that the **Intermediate Value Theorem 1.23.11** lets us construct such a line for a single polygonal region. This corollary lets us do it for two simultaneously.

Proof. Embed $\mathbf{R}^2$ into $\mathbf{R}^2 \times \{1\} \subseteq \mathbf{R}^3$. Given a unit vector $\overline{x} \in S^2$, let $f_i(\overline{x})$ denote the area of the region A_i on the $\overline{x}$-side of the normal plane.

Also note that $f_i(\overline{x}) + f_i(-\overline{x})$ is the area of A_i.

Now let $F = (f_1, f_2) : S^2 \to \mathbf{R}^2$. By **Borsuk-Ulam 1.57.1**, there exists $\overline{x} \in S^2$ such that $F(\overline{x}) = F(-\overline{x})$. This implies that $f_i(\overline{x}) = f_i(-\overline{x})$, so hence $f_i(\overline{x})$ is half the area of A_i, as desired. □

1.58 Section 58: Deformation Retracts and Homotopy Type

Definitions: homotopy equivalence, deformation retraction, contractible
Main Idea: Deformation retractions are important, especially with regards to torsion (summer class).

Definition 1.58.1. A map $f : X \to Y$ is a **homotopy equivalence** if there exists $g : Y \to X$ such that $fg \simeq id_Y$ and $gf \simeq id_X$. In other words, f and g are homeomorphic up to homotopy. We call g the homotopy inverse of f. If a homotopy equivalence exists, we say that X and Y are homotopy equivalent, or have the same homotopy type, and we write $X \simeq Y$.

Lemma 1.58.2. *Homotopy equivalence between spaces is an equivalence relation.*

Proof. Reflexive: $id_X \simeq id_X$

Symmetric: by definition, if g is the homotopy inverse of f, then f is the homotopy inverse of g.

Transitive: it's just path homotopy transitivity (see **Lemma 1.51.6**). □

Lemma 1.58.3. *Let $h, k : X \to Y$ be homotopic via the homotopy $H : X \times I \to Y$, and let $y_0 = h(x_0)$ and $y_1 = k(x_0)$. If $\alpha : I \to Y$ is the path $\alpha(t) = H(x_0, t)$, then $k_* = \hat{\alpha} h_*$; i.e.,*

$$\begin{tikzcd}
\pi_1(X, x_0) \arrow[rr, "h_*"] \arrow[dr, bend right, "k_*"'] & & \pi_1(Y, y_0) \arrow[dl, bend left, "\hat{\alpha}"] \\
& \pi_1(Y, y_1) &
\end{tikzcd}$$

[11] Note in fact that $[h] = [{w_1}^m] \in \pi_1(S^1, 1)$.

commutes.

Proof. We need to show that for a loop f based at $x_0 \in X$, that $k_*([f]) = [kf] = [\overline{\alpha}][hf][\alpha] = \hat{\alpha}(h_*([f]))$. Equivalently, $[\alpha][kf] = [hf][\alpha]$. So let's build a homotopy.

Let $F : I \times I \to X \times I$ be the map $F(s,t) = (f(s),t)$. Then the bottom of $I \times I$, β_0, maps to $(f(s),0)$, which maps to hf. The top of $I \times I$, β_1, maps to $(f(s),1)$, which maps to kf. Then if γ_0 is the right of $I \times I$ mapping to $(f(1),t)$ and γ_1 is the left of $I \times I$ mapping to $(f(0),t)$, then $\beta_0\gamma_0 \mapsto hf\alpha$ and $\gamma_1\beta_1 \mapsto \alpha kf$. As $\beta_0\gamma_0 \simeq \gamma_1\beta_1$, so too $hf\alpha \simeq \alpha kf$. $\square$

Theorem 1.58.4. *Let $f : X \to Y$ be a homotopy equivalence; let $y_0 = f(x_0)$. Then $f_* : \pi_1(X,x_0) \to \pi_1(Y,y_0)$ is an isomorphism.*

Proof. f and g are homeomorphic up to homotopy. Since the fundamental group only sees equivalence classes of paths under homotopy, by **Corollary 1.52.7**, f_* is an isomorphism. That's morally what's going on.

Explicitly, fix $x_0 \in X$. By **Lemma 1.58.3**, there exists a path α from x_0 to $gf(x_0) = x_1$ such that $g_*f_* = (gf)_* = \hat{\alpha}id_{\pi_1(X,x_0)} = \hat{\alpha}$.

The change of base point map $\hat{\alpha}$ is an isomorphism from $\pi_1(X,x_0)$ to $\pi_1(X,x_1)$. Thus, f_* is injective and g_* is surjective.

We remark what f_* and g_* do:

$$\pi_1(X,x_0) \xrightarrow{f_*} \pi_1(Y,y_0 = f(x_0))$$
$$\pi_1(Y,y_0) \xrightarrow{g_*} \pi_1(X,x_1 = g(y_0)).$$

So note in particular where the base points are sent.

By symmetrical argument, we can use a path β from y_0 to $fg(y_0)$ to get that ${f_*}'g_* = \hat{\beta}id_{\pi_1(Y,y_0)} = \hat{\beta}$, and then g_* is injective and ${f_*}'$ is surjective. Note that this is a new ${f_*}'$, since we have

$$\pi_1(X,x_1) \xrightarrow{{f_*}'} \pi_1(Y,y_1 = f(x_1)).$$

However, this is the same g_*. Thus g_* is bijective, and therefore an isomorphism. Then, since $g_*f_* = \hat{\alpha}$, $f_* = (g_*)^{-1}\hat{\alpha}$ is a composition of isomorphisms, hence itself an isomorphism, and we are done. $\square$

Corollary 1.58.5. *If $h : X \to Y$ is nulhomotopic, then h_* is trivial.*

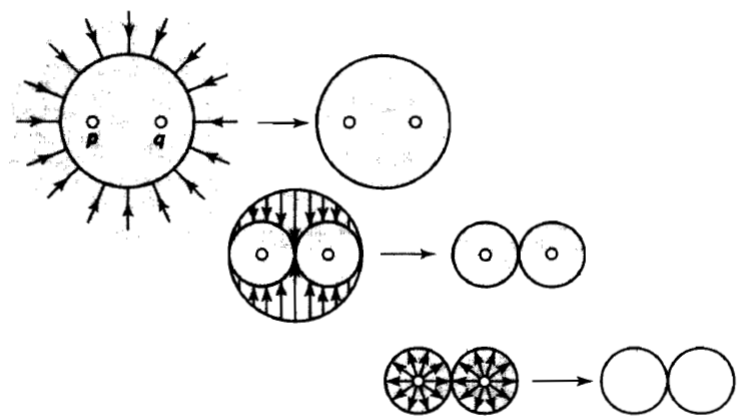
Definition 1.58.6. A subspace $A \subseteq X$ is a **deformation retract** if there exists $H : X \times I \to X$ such that

1. $H(x,0) = x$ for all $x \in X$,
2. $H(a,t) = a$ for all $a \in A$, and
3. $H(x,1) \in A$ for all $x \in X$.

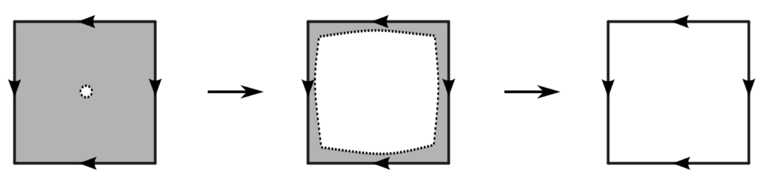
In other words, H is a homotopy between id_X and a retraction $r : X \to A$ that is constant on A.

Example 1.58.7. S^n is a deformation retract on $\mathbf{R}^{n+1} \setminus \{\overline{0}\}$: Let $H(\overline{x}, t) = (1-t)\overline{x} + t\frac{\overline{x}}{\|\overline{x}\|}$.

Example 1.58.8. $\mathbf{R}^2 \setminus \{p, q\}$ deformation retracts onto the figure eight graph and the theta graph. Most often we are vague about describing the explicit maps, so pictures and verbal arguments suffice. Such an example is given below for retracting onto the figure eight.



Example 1.58.9. The space $S^1 \times S^1 \setminus \{p\}$ deformation retracts onto the figure eight graph. This can be seen using the flat torus, an identified polygon, below:



Lemma 1.58.10. *A deformation retraction induces a homotopy equivalence.*

Proof. Let $r : X \to A$ be the retraction, $\iota : A \to X$ the inclusion, and H the homotopy. So $r(x) = H(x, 1)$. Then $r\iota = id_A$ on the nose, since r is a retraction, and running from time 0 to time 1, we have $id_X \simeq \iota r$ by the homotopy H. $\square$

Example 1.58.11. See that $\pi_1(\mathbf{R}^2 \setminus \{\overline{0}\}) \cong \mathbf{Z}$, since $\pi_1(\mathbf{R}^{n+1} \setminus \{\overline{0}\}) \cong \pi_1(S^n)$.

Definition 1.58.12. A space X is contractible if the identity map is homotopic to a constant map.

Lemma 1.58.13. *If X is contractible and $X \simeq Y$, then Y is contractible.*

Proof. Since $X \simeq Y$, we have $f : X \to Y$ and $g : Y \to X$ homotopy inverses. Let $H : X \times I \to X$ be the homotopy from id_X to a constant map $H(x, 1) = x_0$. Then let

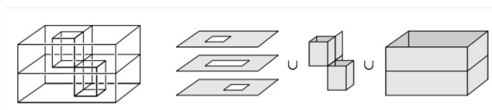
$$K_2 : Y \times I \xrightarrow{g \times id_X} X \times I \xrightarrow{H} X \xrightarrow{f} Y$$

So $K_2(y, t) = f(H(g(y), t))$. Then when $t = 0$, $K_2(y, 0) = fg(y)$, and when $t = 1$, $K_2(y, 1) = f(x_0)$ which is constant. Let $K_1 : Y \times I \to Y$ be a homotopy from id_Y to fg. Then we define

$$K(y,t) = \begin{cases} K_1(y, 2t) & \text{if } 0 \leq t \leq \frac{1}{2}, \\ K_2(y, 2t-1) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

This is a homotopy between id_Y and $f(x_0)$, so Y is contractible, as desired. $\square$

Example 1.58.14. The house with two rooms is contractible, since it is a deformation retract of D^3, but it is not obviously retractible.



1.59 Section 59: The Fundamental Group of S^n

Definitions:
Main Idea: It's trivial if not S^1.

Theorem 1.59.1. *If $n \geq 2$, then $\pi_1(S^n) = \{1\}$.*

Proof. Let $f : I \to S^n$ be a loop based at $x_0 \in S^n$. We need to show $[f] = 1$. See that if there exists $p \in S^n \setminus f(I)$, then using stereographic projection, there is a homeomorphism $h : S^n \setminus \{p\} \to \mathbf{R}^n$, and so as $\mathbf{R}^n$ is contractible, $f \simeq e_{x_0}$. So it suffices to homotope f to miss a point. This isn't necessarily trivial, as **Section 44: A Space-Filling Curve** has shown us.

So, we proceed. Fix $p \in S^n$ where $p \neq x_0$. Let U be a neighborhood of p such that $\overline{U}$ is homeomorphic to D^n and $x_0 \notin \overline{U}$. Then $f^{-1}(U)$ is an open set in $(0, 1)$, hence a union of countably many disjoint intervals (a_i, b_i). Note further that $f^{-1}(p)$ is compact, as it is closed and bounded, and so contained in only finitely many of the intervals (a_i, b_i). On such an interval $[a_i, b_i]$, homotope $f|_{[a_i,b_i]} : [a_i, b_i] \to \overline{U}$ so that the image is contained on the boundary $\partial U = \partial D^n$. As D^n is simply connected, any two paths between $f(a_i)$ and $f(b_i)$ are homotopic. Repeat on all closed intervals as necessary. $\square$

1.60 Section 60: The Fundamental Groups of Some Surfaces

Definitions:

Main Idea: The fundamental group of a product, the fundamental group of the torus, the fundamental group of $\mathbf{RP}^2$, and the fundamental group of the Klein bottle, among others.

Theorem 1.60.1. $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$.

Proof. We have that $p : X \times Y \to X$ and $q : X \times Y \to Y$, p and q projection maps, define a homomorphism $\Phi : \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0) \times \pi_1(Y, y_0)$ by $\Phi([f]) = (p_*([f]), q_*([f]))$. We show that Φ is an isomorphism, by showing it is bijective.

To see surjective, given paths $g : I \to X$, $h : I \to Y$, define the path $f = (g, h) : I \to X \times Y$. As $g = pf$ and $h = qf$, then $\Phi([f]) = ([g], [h])$.

To see injective, suppose $\Phi([f]) = 1$. We show $[f]$ is nulhomotopic. Then $pf \simeq e_{x_0}$ via a homotopy G and $qf \simeq e_{y_0}$ via H. Then the homotopy $F : I \times I \to X \times Y$ defined by $F(s, t) = (G(s, t), H(s, t))$ is a homotopy between f and $e_{(x_0, y_0)}$.

Thus Φ is an isomorphism. $\square$

Example 1.60.2. $\pi_1(S^1 \times S^1) \cong \mathbf{Z} \times \mathbf{Z} = \mathbf{Z}^2 = \langle [\begin{smallmatrix} a \\ b \end{smallmatrix}] \mid a, b \in \mathbf{Z} \rangle = \langle a, b \mid ab = ba \rangle$.

π_1 is generated by two loops, one around the torus and one through the middle, a and b. The generators commute and are path homotopic, as

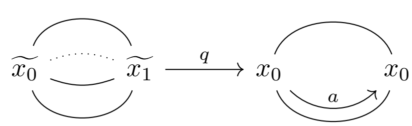
$$\begin{tikzcd} * \arrow[r, "a"] & * \\ * \arrow[u, "b"] \arrow[r, "a"'] \arrow[ur, bend left=60, "ba"] & * \arrow[u, "b"'] \arrow[from=2-1, to=1-2, bend right=60, "ab"'] \end{tikzcd}$$

Thus $ab \simeq ba$.

Example 1.60.3. Consider the real projective plane $\mathbf{RP}^2 = {S^2}/{\overline{x} \sim -\overline{x}}$.

The quotient map $q : S^2 \to \mathbf{RP}^2$ is a covering map. We know that S^2 is simply connected, so the lifting correspondence map (**Definition 1.54.5**) $\varphi : \pi_1(\mathbf{RP}^2, x_0) \to q^{-1}(x_0)$ is a bijection.

Since $q^{-1}(x_0)$ only has two elements $\widetilde{x_0}$ and $\widetilde{x_1}$, $\pi_1(\mathbf{RP}^2, x_0) \cong {\mathbf{Z}}/{2\mathbf{Z}} = \mathbf{Z}_2$.



$$S^2 \xrightarrow{q} \mathbf{RP}^2$$

The loop a is a generator for $\pi_1(\mathbf{RP}^2, x_0)$, where a is 1 and a^2 is 0.

Lemma 1.60.4. *Let ∞ be the figure eight graph. Then $\pi_1(\infty)$ is non-abelian.*

Proof. Consider the covering space pictured:

$$
\begin{tikzcd}
\widetilde{\infty}: \arrow[d, "p"] & \widetilde{x_1} \arrow[loop left, "a"] \arrow[r, bend left, "b"] & \widetilde{x_0} \arrow[l, bend left, "b"] \arrow[r, bend right, "a"'] \arrow[d, "p"] & \widetilde{x_2} \arrow[l, bend right, "a"'] \arrow[loop right, "b"] \\
\infty: & & x_0 \arrow[loop left, "a"] \arrow[loop right, "b"] &
\end{tikzcd}
$$

Let $a : I \to \infty$ be the loop transversing the a edge. Similarly define $b : I \to \infty$. Then ab is a loop based at x_0, whose lift starts at $\widetilde{x_0}$ and ends at $\widetilde{x_2}$. Further, ba is a loop, whose lift starts at $\widetilde{x_0}$ and ends at $\widetilde{x_1}$. In other words, the lifted paths of ab and ba starting at $\widetilde{x_0}$ satisfy $\widetilde{ab}(1) = \widetilde{x_2} \neq \widetilde{x_1} = \widetilde{ba}(1)$.

We've shown that homotopic paths in the base space lift to the same endpoints (**Corollary 1.54.4**). Therefore, $ab \not\simeq ba$, and $\pi_1(\infty)$ is not abelian. $\square$

Theorem 1.60.5. *Let S_2 be the surfact of genus 2, a two-holed torus. Then $\pi_1(S_2)$ is non-abelian.*

Proof. There exists a retraction $r : S_2 \to \infty$, where ∞ is the figure eight graph. This is obvious.

Thus $r_* : \pi_1(S_2) \to \pi_1(\infty)$ is surjective, by **Lemma 1.55.4**. Since $\pi_1(\infty)$ is non-abelian by **Lemma 1.60.4**, $\pi_1(S_2)$ is non-abelian, as claimed. $\square$

Theorem 1.60.6. *Let K be the Klein bottle. Then $\pi_1(K)$ is non-abelian.*

Proof. We do the same strategy as **Lemma 1.60.4**; we find a suitable cover. Consider:

$$
\begin{tikzcd}
\widetilde{K}: \arrow[dddddd, "p"] & \widetilde{x_1} \arrow[r, "a"] & \widetilde{x_1} \arrow[d, "b"] \\
& \widetilde{x_2} \arrow[r, "a"] \arrow[u, "b"] & \widetilde{x_0} \arrow[d, "b"] \\
& \widetilde{x_0} \arrow[r, "a"] \arrow[u, "b"] & \widetilde{x_2} \arrow[d, "b"] \\
& \widetilde{x_1} \arrow[r, "a"] \arrow[u, "b"] \arrow[d, "p"] & \widetilde{x_1} \\
& x_0 \arrow[r, "a"] & x_0 \arrow[d, "b"] \\
K: & x_0 \arrow[r, "a"] \arrow[u, "b"] & x_0
\end{tikzcd}
$$

Recognize now that $\widetilde{K}$ is clearly also a Klein bottle.

Now, take ba a loop in K and lift it to a path starting at $\widetilde{x_0}$. Then it ends at $\widetilde{x_0}$, so $\widetilde{ba}(1) = \widetilde{x_0}$. But ab, also a loop in K, lifted, ends at $\widetilde{x_1}$; i.e., $\widetilde{ab}(1) = \widetilde{x_1}$. These are distinct in the covering space, and again we see that the fundamental group $\pi_1(K)$ is non-commutative/non-abelian. $\square$

Proposition 1.60.7. $\pi_1(K) = \langle a, b \mid ab^{-1} = ba \rangle$.

Theorem 1.60.8. *Suppose n is 1 or 2. Then $\mathbf{R}^n$ is homeomorphic to $\mathbf{R}^m$ if and only if $m = n$.*

We remark that this is true for all n, but we don't yet have the machinery to show this.

Proof. Suppose $h : \mathbf{R}^n \to \mathbf{R}^m$ is a homeomorphism. Then h restricts to a homeomorphism $h|_{\mathbf{R}^n \setminus \{\bar{0}\}} : \mathbf{R}^n \setminus \{\bar{0}\} \to \mathbf{R}^m \setminus \{h(\bar{0})\}$, which can be recognized as a map $S^{n-1} \to S^{m-1}$.

Then S^{n-1} is homotopy equivalent to S^{m-1}. If $n = 1$, then $S^0 = \{a, b\}$ is not connected, and hence not homotopy equivalent to S^{m-1} for $m > 1$. If $n = 2$, then $\pi_1(S^1) \cong \mathbf{Z}$ and hence S^1 is not homotopy equivalent to S^{m-1} for $m > 2$. Whether by disconnectedness or different fundamental groups, we see that if n is 1 or 2, $\mathbf{R}^n \cong \mathbf{R}^m$ if and only if $m = n$. □

2 Topology II

The notes for this semester are a lot harder to organize, because Hatcher is a bad book. I have elected to do so to the best of my judgement and ability, which means we don't really get the continuity of theme that we do in the sections in semester one. I'll do my best to make up for this with good exposition.

The outline for this semester can be vaguely characterized by the following topics:

1. CW-complexes, a combinatorial way to build a topological space,
2. exploring the fundamental group π_1 further; Van Kampen's theorem, classification of covering spaces, etc. in particular,
3. homology H_*, another topological invariant related to π_1,
4. cohomology H^*, related to H_*,
5. Poincaré duality, and
6. the classification of surfaces, higher homotopy groups π_n, and other topics.

2.1 Section 1: CW-complexes

Definitions: n-cell, CW-complex, CW-pair
Main Idea: We define a CW-complex and list a few examples.

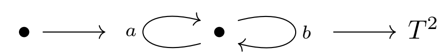
We begin with some underlying geometric notions. We can build surfaces in the following manner:

1. Start with a vertex, considered as D^0.

2. Attach edges $a, b, c, \ldots$ (thought of as D^1) so that their boundaries are D^0.

3. Attach a square/disk so that its boundary is D^1.

Example 2.1.1. Consider the torus, which we will write as $T = T^2 = S^1 \times S^1$. We can start with a single vertex D^0, then attach two edges a and b, D^1s, and then fill in the surface with a single D^2.



This can be suitably generalized beyond just surfaces.

Definition 2.1.2. We define an n-**cell** D^n to be a space homeomorphic to $\overline{B_d(\overline{0}, 1)} \subseteq \mathbf{R}^n$, the closed unit ball in $\mathbf{R}^n$.

Given an n-cell D^n, the boundary ∂D^n is the image of $S^{n-1} \subseteq \overline{B_d(\overline{0}, 1)}$ under the homeomorphism, i.e., $\partial D^n \cong S^{n-1}$.

Finally, an open n-cell is $e^n = D^n \setminus \partial D^n$.

Definition 2.1.3. We define a CW-**complex** inductively, describing a structure of n-skeletons:

Begin with a 0-skeleton, X^0, the disjoint union of 0-cells.

For the 1-skeleton, for each 1-cell ${D^1}_\alpha$, define an attaching map $\varphi_\alpha : \partial {D^1}_\alpha \to X^0$. Then the 1-skeleton is $X^0 \cup \bigcup_\alpha {D^1}_\alpha \big/ x \sim \varphi_\alpha(x)$ for all $x \in \partial {D^1}_\alpha$. As a set, $X^1 = X^0 \amalg \coprod_\alpha {e^1}_\alpha$, where ${e^1}_\alpha$ is an open 1-cell ${D^1}_\alpha \setminus \partial {D^1}_\alpha$.

Now, to define the n-skeleton, for each n-cell ${D^n}_\alpha$, there exists an attaching map $\varphi_\alpha : \partial {D^n}_\alpha \to X^{n-1}$. Then

$$X^n = X^{n-1} \cup \bigcup_\alpha {D^n}_\alpha \big/ x \sim \varphi_\alpha(x)$$

for all $x \in \partial {D^n}_\alpha$. Again as a set,

$$\begin{aligned} X^n &= \amalg \coprod_\alpha {e^n}_\alpha \\ &= \coprod_{i=0}^{n} {e^i}_\alpha. \end{aligned}$$

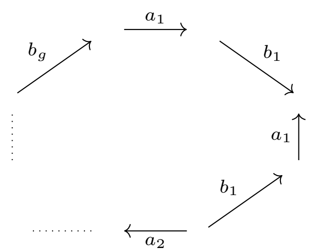
The CW-complex is $X = \cup X^n$. The topology on X is that a set $U \subseteq X$ is open if and only if $U \cap X^n$ is open in X^n for all n.

Example 2.1.4. To see an explicit example of a 1-skeleton construction, let X^0 be a single 0-cell e^0, and then define the attaching map $\varphi : \partial D^1 \to X^0$ to be the constant map. Then X^1 is

i.e., homeomorphic to S^1. We'll see other ways to build S^1 as a CW-complex.

Example 2.1.5. Any 1-dimensional CW-complex is just a graph, where the vertices are 0-cells, and the edges 1-cells.

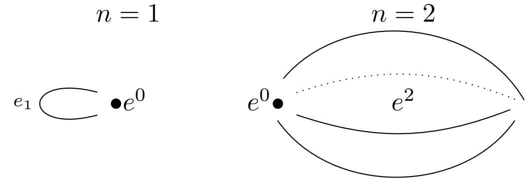
Example 2.1.6. A surface of genus g, S_g, is a CW-complex with one 0-cell, $2g$ 1-cells, $(a_1, b_1, a_2, b_2, ..., a_g, b_g)$, and one 2-cell.



Example 2.1.7. An n dimensional sphere S^n is a CW-complex in two different ways:

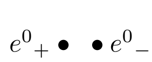
1. One 0-cell and one n-cell, attached via the constant map $\partial D^n \cong S^{n-1} \xrightarrow{\varphi} X^0$; i.e.,

$n = 1$ $\qquad$ $n = 2$

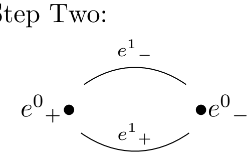


2. Two i-cells, ${D^i}_+$ and ${D^i}_-$, for $0 \leq i \leq n$. Then $X^i \cong S^i$ and the attaching maps are $\varphi_\pm : \partial {D^{i+1}}_\pm \cong S^i \to X^i$ are homeomorphisms. We show the construction in steps:

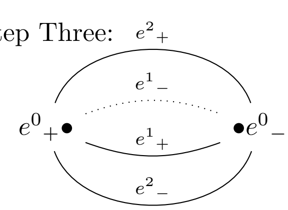
Step One:



Step Two:



Step Three:



We remark now and prove later that the Euler characteristic, **Definition 2.9.1**, is invariant between construction of CW-complexes.

Using the second CW-structure on S^n, we can now define $S^\infty = \bigcup_n S^n$.

Example 2.1.8. Note that the equivalence relation $\overline{x} \sim -\overline{x}$ respects the cell structure in part two of **Example 2.1.7**. Therefore, $\mathbf{RP}^n = {S^n}\big/{\overline{x} \sim -\overline{x}} = e^0 \cup e^1 \cup e^2 \cup ... \cup e^n$. We have attaching maps $\partial D^{i+1} \cong S^i \xrightarrow{\varphi_{i+1}} X^i \cong \mathbf{RP}^i$ which are the usual quotient maps of identifying antipodes.

We can therefore also define $\mathbf{RP}^\infty = {S^\infty}\big/{\overline{x} \sim -\overline{x}} = \bigcup_n \mathbf{RP}^n$.

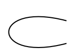
Definition 2.1.9. A CW**-pair** is a pair (X, A) where X is a CW-complex and A is a subcomplex, i.e., A is closed in X and a union of cells.

Example 2.1.10. We remark that every n-skeleton X^n is a subcomplex in X, so (X, X^n) is a CW-pair.

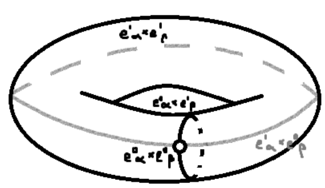
2.2 Section 2: Operations on CW-complexes

Definitions: CW-product, CW-quotient, suspension, wedge sum, cellular, attaching space, mapping cylinder
Main Idea: Description of some things you can do to CW-complexes. We also get homotopy equivalences in special circumstances, which are nice.

Definition 2.2.1. If X, and Y are CW-complexes, then so is the **product of** CW**-complexes** $X \times Y$. The CW-structure on $X \times Y$ is given by its open cells, which are ${e^m}_\alpha \times {e^n}_\beta$ where ${e^m}_\alpha$ is an open cell in X and ${e^n}_\beta$ is an open cell in Y.

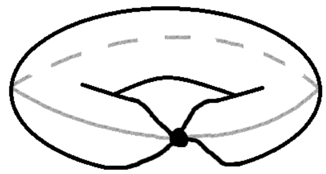
Example 2.2.2. Let ${S^1}_\alpha$ be $e_{1\alpha}$  $\bullet {e^0}_\alpha$ and let ${S^1}_\beta$ be $e_{1\beta}$  $\bullet {e^0}_\beta$.

Then we see that the torus ${S^1}_\alpha \times {S^1}_\beta$ has one 0-cell, ${e^0}_\alpha \times {e^0}_\beta$, two 1-cells, ${e^1}_\alpha \times {e^0}_\beta$ and ${e^0}_\alpha \times {e^1}_\beta$, and one 2-cell, ${e^1}_\alpha \times {e^1}_\beta$.



Definition 2.2.3. If (X, A) is a CW-pair, then we can define the **quotient of** CW**-complexes** by ${X}\big/{A}$, a CW-complex, where the open cells are exactly the open sets in $X \setminus A$, and one 0-cell for A. The intuition built up for quotient spaces/groups/etc is exactly this: take the subspace A and collapse it to a point.

Example 2.2.4. Consider the CW structure on $S^1 \times S^1$ from **Example 2.2.2**. Let $A = e^0{}_\alpha \times e^1{}_\beta$. Then ${}^{X}\!/_{A}$ is pictured below.

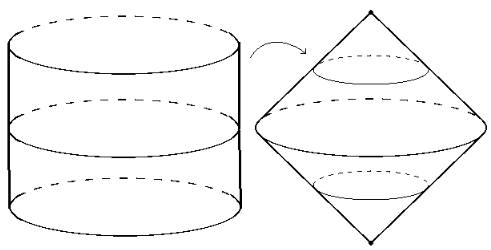


Definition 2.2.5. We define the **suspension** on a space X to be

$$SX = {}^{X \times I}\!\Big/_{\substack{(x,0)\sim(x',0)\\(x,1)\sim(x',1)}}$$

for all $x, x' \in X$.

Example 2.2.6. The motivating example is SS^1; everything is completely described in the following picture.

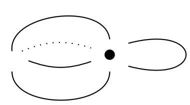


In general, $SS^n = S^{n+1}$.

Definition 2.2.7. Given $x_0 \in X$ and $y_0 \in Y$, we define the **wedge sum** $X \vee Y = {}^{X \amalg Y}\!/_{x_0 \sim y_0}$.

Example 2.2.8. $S^1 \vee S^1 = \infty$, the figure eight graph.

Example 2.2.9. $S^1 \vee S^2$ is a sphere with an arc as a handle:

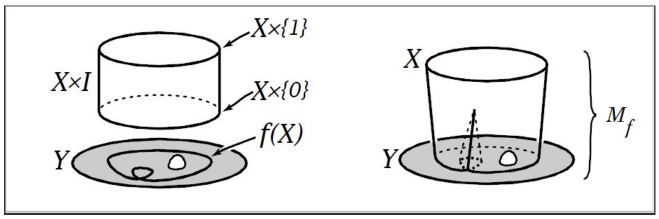


Definition 2.2.10. Let X and Y be CW-complexes. We say that a map $f : X \to Y$ is **cellular** if $f(X^n) \subseteq Y^n$.

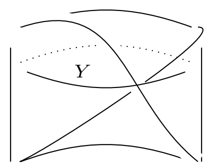
Definition 2.2.11. If (X, A) is a CW-pair, Y is a CW-complex, and $f : A \to Y$ is cellular, then we define the **attaching space** $X \sqcup_f Y = {}^{X \amalg Y}\!/_{a \sim f(a)}$ for all $a \in A$.

We remark that $X \sqcup_f Y$ is a CW-complex.

Definition 2.2.12. If X and Y are CW-complexes, $A = X \times \{0\} \subseteq X \times I$, and $g : X \to Y$ is a cellular map, then we can think of f as a map $A \to Y$. A special type of attaching space, called a **mapping cylinder**, is defined to be $M_f = X \times I \sqcup_f Y$.



Example 2.2.13. Let $f : S^1 \to S^1$ where $z \mapsto z^2$. Then M_f is the Möbius band. The codomain, $Y = S^1$, is the center circle.

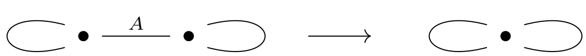


Recall **Definition 1.58.1**, which says that two spaces X and Y are homotopy equivalent if there exist maps f and g that are homotopic inverses. We now give two criteria for $X \simeq Y$:

Proposition 2.2.14. *If (X, A) is a CW-pair and A is contractible, then $X \to {}^{X}\!/_{A}$ is a homotopy equivalence.*

Example 2.2.15. Let X be a graph and A be a contractible subgraph (i.e., a subtree). Then collapsing the subtree to a point (i.e., the map $X \to {}^{X}\!/_{A}$, is a homotopy equivalence.

$$X \longrightarrow {}^{X}\!/_{A}$$

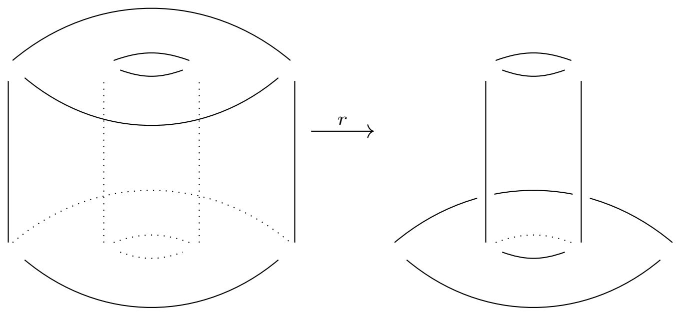


Proposition 2.2.16. *If (X, A) is a CW-pair, Y is a CW-complex, and $f, g : A \to Y$ are homotopic, then $X \sqcup_f Y \simeq X \sqcup_g Y$. In other words, we can be vague about describing explicit attaching maps.*

Theorem 2.2.17 (**Homotopy Extension Property**)**.** *If (X, A) is a CW-pair, Y is a CW-complex, $F : X \to Y$ is a map, and $f_t : A \to Y$ is a homotopy of $F|_A$, then there exists a homotopy $F_t : X \to Y$ such that $F_t|_A = f_t$.*

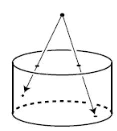
Proof. We first show that $X \times \{0\} \cup A \times I$ is a deformation retract (**Definition 1.58.6**) of $X \times I$.

Let $r : X \times I \to X \times \{0\} \cup A \times I$ be a retraction. Then we set $F_t = f_t r$. The retraction is evident; simply push on the top of $X \times I$, leaving $A \times I$ and $X \times \{0\}$ intact.



We give it explicitly, though. There exists a retraction $r' : D^n \times I \to D^n \times \{0\} \cup \partial D^n \times I$ by radial projection from $(\overline{0}, 2) \in \mathbf{R}^n \times I$.

For example, if $n = 2$, the picture is almost exactly as we already have it:



The retraction r' is explicitly $r'_t(x) = tr'(x) + (1-t)x$.

This provides a deformation retraction of $X^n \times I$ onto $X^n \times \{0\} \cup (X^{n-1} \cup A^n) \times I$.

To retract not only an n-skeleton but all of X, we can perform this retraction during the time interval $\left[\frac{1}{2^{n+1}}, \frac{1}{2^n}\right]$. This gives an infinite concatenation of homotopies, which is the deformation retraction we seek, as long as we can show it is continuous.

But indeed it is, because the homotopy is stationary on $X^n \times I$ during $\left[0, \frac{1}{2^{n+1}}\right]$, so it is continuous on $X^n \times I$ for any n. Since the topology of a CW-complex defines open cells as open on every n-skeleton, this suffices. $\square$

2.3 Section 3: Discussion of Algebra

Definitions: free product, extension of homomorphism
Main Idea: Brush up on your algebra before Van Kampen.

Definition 2.3.1. Let $\{G_\alpha\}_{\alpha \in J}$ be a family of groups. We define the **free product** $\underset{\alpha}{*} G_\alpha$: its elements are reduced words $g_1 g_2 ... g_m$ where if $m \neq 1$, then $g_i \in G_{\alpha_i} \setminus \{e\}$, and $\alpha_i \neq \alpha_{i+1}$. In other words, these are reduced words, where if the length is not 1, we can reduce by removing identities e or by performing a group operation if adjacent elements are in the same group G_{α_i}. Its operation is concatenation and reduction.

Proposition 2.3.2 (**Universal Property**)**.** *The free product satisifes the* ***Universal Property****:*

For groups $\{G_\alpha\}_{\alpha \in J}$, $\underset{\alpha}{*} G_\alpha$, and H, there exist inclusion maps $\iota_\alpha : G_\alpha \to \underset{\alpha}{*} G_\alpha$ such that for any collection of homomorphisms $f_\alpha : G_\alpha \to H$, there exists a unique homomorphism $f : \underset{\alpha}{*} G_\alpha \to H$ such that $f_\alpha = f\iota_\alpha$. In other words, there exists a unique f such that the following diagram commutes:

$$\begin{tikzcd} G_\alpha \arrow[rr, "\iota_\alpha"] \arrow[dr, "f_\alpha"'] & & \underset{\alpha}{*} G_\alpha \arrow[dl, dotted, "f"] \\ & H & \end{tikzcd}$$

Example 2.3.3. We give an example of a free group: $\mathbf{Z} * \mathbf{Z} = \langle a \rangle * \langle b \rangle$. This is the free group of rank two, $\mathbf{F}_2$. A reduced word is something like $a^2b^{-3}a^4b^{-2}$, while a non-reduced word is something like $a^{-2}ab$, which reduces to $a^{-1}b$. To see the operation, $(a^2ba)(a^{-1}b^{-1}a) = a^2baa^{-1}b^{-1}a = a^3$.

Note that $\mathbf{Z} * \mathbf{Z}$ is not abeliean, versus $\mathbf{Z}^2 = \mathbf{Z} \times \mathbf{Z} = \mathbf{Z} \oplus \mathbf{Z}$, which is abelian.

Example 2.3.4. For another example, $\mathbf{Z}_2 * \mathbf{Z}_2 = \langle a \rangle * \langle b \rangle$. Then elements must be alternating words in a and b, else they are not reduced words (since $a^2 = 1$, $b^2 = 1$). Note that ab has infinite order, since $(ab)^n = ababab...ab$, n times.

We remark that $|G * H| = \infty$ always.

Definition 2.3.5. Given $g = g_1...g_m \in \underset{\alpha}{*} G_\alpha$ where $g_i \in G_{\alpha_i}$, we can define a homomorphism $f : \underset{\alpha}{*} G_\alpha \to H$ which is an **extension of the homomorphisms** $f_\alpha : G_\alpha \to H$, via $f(g) = f_{\alpha_1}(g_1)f_{\alpha_2}(g_2)...f_{\alpha_m}(g_m)$. This is just the **Universal Property 2.3.2** in action.

Example 2.3.6. In $\mathbf{Z} * \mathbf{Z} = \langle a \rangle * \langle b \rangle$, let $f_{1,2} : \mathbf{Z} \to \mathbf{Z}$ be homomorphisms, i.e., $f_1(a) = 1$ and $f_2(b) = 1$, since a homomorphism in $\mathbf{Z}$ is uniquely determined by where it sends the generator.

Then $f : \mathbf{Z} * \mathbf{Z} \to \mathbf{Z}$. Some examples: $f(a^2b^{-1}a^{-1}) = 2 - 1 - 1 = 0$, and $f(ab^2ab^{-1}) = 1 + 2 + 1 - 1 = 3$.

2.4 Section 4: Van Kampen's Theorem

Definitions: normal closure, group presentation
Main Idea: Sit with this section for a while to get more comfortable with it. Group presentations and **Corollary 2.4.4** in particular are good places to start, but make sure you have everything.

The main idea behind **Van Kampen's Theorem 2.4.3** is that we can compute the fundamental group of a space by decomposing it into subspaces whose fundamental groups are known.

Definition 2.4.1. If G is a group and $S \subseteq G$ a subset, then the **normal closure of** S is the normal subgroup $\langle\langle S\rangle\rangle$, which is defined to be the intersection of all normal subgroups containing S, or equivalently, $\{\prod r_i {s_i}^{\pm 1} {r_i}^{-1} \mid r_i \in G, s_i \in S\}$.

Example 2.4.2. If $S = \{[a,b]\}$, $a, b \in \langle a\rangle * \langle b\rangle = \mathbf{Z} * \mathbf{Z}$, then one can verify that $\langle\langle [a,b]\rangle\rangle$ is the collection of words whose a-exponents sum to 0 and b-exponents sum to 0.

For example, $a^2b^2a^{-1}ba^{-1}b^{-3} \in \langle\langle [a,b]\rangle\rangle$.

Theorem 2.4.3 (**Van Kampen's Theorem**). *Suppose* $X = \bigcup_\alpha A_\alpha$*, where* A_α *is open and path connected, and* $x_0 \in A_\alpha$ *for all* α*. Then:*

If $A_\alpha \cap B_\alpha$ *are path connected for all pairs* α, β*, then* $\varphi : \ast_\alpha \pi_1(A_\alpha, x_0) \to \pi_1(X, x_0)$ *is surjective.*

If, in addition, $A_\alpha \cap A_\beta \cap A_\gamma$ *are path connected for all triples* α, β, γ*, then* $\ker\varphi = \langle\langle \iota_{\alpha\beta}(w)\iota_{\beta\alpha}(w)^{-1} \mid w \in \pi_1(A_\alpha \cap A_\beta), \iota_{\alpha\beta}(w) \in \pi_1(A_\alpha), \iota_{\beta\alpha}(w)^{-1} \in \pi_1(A_\beta)\rangle\rangle$.

Ultimately, we conclude that φ *induces an isomorphism*

$$\pi_1(X) \cong {\ast_\alpha \pi_1(A_\alpha)}\big/{\ker\varphi}.$$

We describe the notation used:

If $A_\alpha \hookrightarrow X$ is inclusion, then there is an induced homomorphism $j_\alpha : \pi_1(A_\alpha) \to \pi_1(X)$.

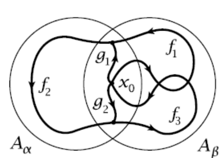
The map φ is just the extension, **Definition 2.3.5**, to the free product.

If $\iota : A_\alpha \cap A_\beta \hookrightarrow A_\alpha$ is inclusion, then there is an induced $\iota_{\alpha\beta} : \pi_1(A_\alpha \cap A_\beta) \to \pi_1(A_\alpha)$.

We remark that if w is a loop in $\pi_1(A_\alpha \cap A_{beta})$, then $\varphi(\iota_{\alpha\beta}(w)) = j_\alpha\iota_{\alpha\beta}(w) = j_\beta\iota_{\beta\alpha}(w) = \varphi(\iota_{\beta\alpha}(w))$, which implies that $\iota_{\alpha\beta}(w)\iota_{\beta\alpha}(w)^{-1} \in \ker\varphi$, which is good.

Proof of Van Kampen. First, we show that φ is surjective. If f is a loop in $A_\alpha \cap A_\beta$, then we show that f can be decomposed as a product of loops in each contained in either A_α or A_β.

Let f be a loop in the intersection. Then we break f into three pieces, $f_1f_2f_3$, and then there exist paths g_1 and g_2 remaining entirely in the intersection such that $f_1f_2f_3 \simeq f_1\overline{g_1}g_1f_2\overline{g_2}g_2f_3$, and $f_1\overline{g_1} \in \pi_1(A_\beta)$, $g_1f_2\overline{g_2} \in \pi_1(A_\alpha)$, and $g_2f_3 \in \pi_1(A_\beta)$. See the picture below.



Now we show that $\ker\varphi$ is exactly the normal subgroup described. To do this, we introduce loop factorizations: let $[f]$, an equivalence class of a loop in X, factor into $[f_1][f_2]...[f_k]$, where each $[f_i] \in \pi_1(A_{\alpha_i})$. Factorization is therefore an element of $\mathop{*}\limits_{\alpha} \pi_1(A_\alpha)$. We make three remarks on factorization which are pertinent:

1. Since φ is surjective, every $[f]$ factors.

2. The factorization of $[f]$ is equivalent to the factorization of $[f]'$ if we can combine $[f_i][f_{i+1}]$ into $[f_i f_{i+1}]$ if $\alpha_i = \alpha_{i+1}$. This is because both $[f_i]$ and $[f_{i+1}]$ lie in $\pi_1(A_{\alpha_i})$, so their composition is a loop in A_{α_i}. This won't change the element in $\mathop{*}\limits_{\alpha} \pi_1(A_\alpha)$.

3. We can also recognize equivalent factorizations by regarding $[f_i] \in \pi_1(A_{\alpha_i})$ as an element of $\pi_1(A_{\alpha_{i'}})$ if the loop f_i is in $A_{\alpha_i} \cap A_{\alpha_{i'}}$. This changes the word in $\mathop{*}\limits_{\alpha} \pi_1(A_\alpha)$, but when we quotient by $\ker\varphi$, the factorizations are indeed equivalent.

This shows that equivalent factorizations give the same element in $\mathop{*}\limits_{\alpha} \pi_1(A_\alpha)\big/\ker\varphi$. We just need to show now that any two factorizations of $[f]$ are equivalent. That will show that they are the same element in $\mathop{*}\limits_{\alpha} \pi_1(A_\alpha)\big/\ker\varphi$.

The proof of such a thing is deferred, but it can be done. $\square$

Corollary 2.4.4. *Let X_α be a collection of path connected spaces with base points $x_\alpha \in X_\alpha$ such that x_α is the deformation retraction of some neighborhood $U_\alpha \subseteq X_\alpha$. Then*

$$\pi_1\left(\bigvee X_\alpha, x_0\right) \cong \mathop{*}\limits_{\alpha} \pi_1(X_\alpha, x_\alpha).$$

Proof. Let $A_\alpha = X_\alpha \vee \bigvee\limits_{\beta\neq\alpha} U_\beta$. Then A_α deformation retracts onto X_α, so $\pi_1(X_\alpha) \cong \pi_1(A_\alpha)$. Since $A_\alpha \cap A_\beta = \bigvee\limits_{\gamma\neq\beta,\alpha} U_\gamma$ deformation retracts onto X_α; this implies $\ker\varphi = \{1\}$. Thus by **Van Kampen 2.4.3**, $\pi_1\left(\bigvee X_\alpha, x_0\right) \cong *_\alpha\, \pi_1(X_\alpha, x_\alpha)$, as claimed. $\square$

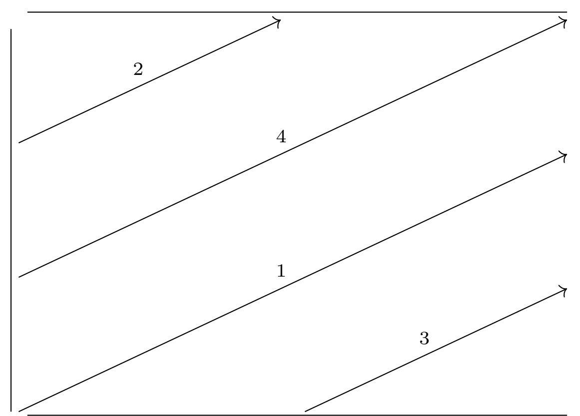
This is the real powerhouse theorem for computing fundamental groups, since we only occasionally discussed normal subgroups. In fact, we later use this to realize every group as the fundamental group of a 2-dimensional CW-complex.

Example 2.4.5. $\pi_1(S^1 \vee S^1) \cong \mathbf{Z} * \mathbf{Z}$.

To see this, let the wedge happen at $\{1\}$ on each S^1. Then let $A_1 = S^1 \vee (S^1 \setminus \{-1\})$ and let $A_2 = (S^1 \setminus \{-1\}) \vee S^1$. Then A_1 deformation retracts to $S^1 \vee \{1\} \cong S^1$, and A_2 deformation retracts to $\{1\} \vee S^1 \cong S^1$. Then $\pi_1(S^1 \vee S^1) \cong \pi_1(S^1) * \pi_1(S^1) = \mathbf{Z} * \mathbf{Z}$.

Example 2.4.6. Consider a Torus knot. If $m, n \in \mathbf{N}$ are relatively prime, then $K = K_{m,n}$ is the image of the embedding $f : S^1 \to S^1 \times S^1 \subseteq \mathbf{R}^3$, where $z \mapsto (z^m, z^n)$.

On a square torus, this is a line of slope $\frac{n}{m}$:



Pictured above is a $(2, 3)$ torus knot, the trefoil.

Now we want to compute $\pi_1(\mathbf{R}^3 \setminus K)$, the fundamental group of the knot complement of some knot K. See that $S^3 \setminus K = \mathbf{R}^3 \setminus K \cup B$, where B is an open ball about ∞. Then we set up the hypotheses of **Van Kampen's Theorem 2.4.3**: $B \cap \mathbf{R}^3 \setminus K$ is homeomorphic to an open ball minus a point which is homeomorphic to $S^2 \times (0, 1)$, which is simply connected.

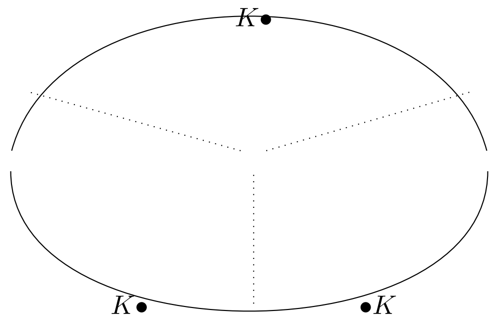
Therefore, by **Van Kampen 2.4.3**,

$$\begin{aligned} \pi_1(S^3 \setminus K) &\cong \pi_1(\mathbf{R}^3 \setminus K) * \pi_1(B) \\ &= \pi_1(\mathbf{R}^3 \setminus K). \end{aligned}$$

So we can work in S^3 and consider the knot complement as $S^3 \setminus K$ instead, which we will find to be easier.

In particular, we can decompose S^3 via a Heegaard splitting: $S^3 = \partial(D^4) = \partial(D^2 \times D^2) = \partial D^2 \times D^2 \cup D^2 \times \partial D^2 = S^1 \times D^2 \cup D^2 \times S^1$. Note that these are two solid tori, and indeed, S^3 is the union of two solid tori glued along their boundary. To picture this, first anchor one solid torus in $\mathbf{R}^3 = S^3 \setminus \{\infty\}$. This anchored torus is the hole of the second solid torus; it passes through the hole of the first, up and out towards infinity, through infinity, and then back up from the bottom to meet itself.

Now, consider the knot K on the surface of the first solid torus. K meets $\{x\} \times D^2 \subseteq S^1 \times D^2$ in exactly m points. Consider when $m = 3$; then $\{x\} \times D^2$, the cross section of the torus, is shown below:



On this cross section, the complement of K deformation retracts onto the dotted "Y." Then let X_m be the mapping cylinder for $x \mapsto x^m$ where X_m is the "Y"s for all x around the torus. We can do the same for X_n in the second solid torus. We get a deformation retraction of the knot complement onto X, defined to be

$$X = X_m \cup X_n = {S^1 \times [0,1]} \Big/ {\substack{(z,0)\sim(z^m,0) \\ (z,1)\sim(z^n,1)}}$$

Since X_m is a mapping cylinder, it deformation retracts onto S^1, so $\pi_1(X_m) \cong \mathbf{Z} = \langle a \rangle$, and similarly $\pi_1(X_n) \cong \mathbf{Z} = \langle b \rangle$. And the intersection $X_m \cap X_n$ is homeomorphic to $S^1 \times (-1,1)$; therefore $\pi_1(X_m \cap X_n) \cong \mathbf{Z}$.

Now we consider the induced maps:

$$\begin{aligned} {\iota_m}^* \pi_1(X_m \cap X_n) &= \langle a^m \rangle \subseteq \pi_1(X_m), \text{ and} \\ {\iota_n}^* \pi_1(X_m \cap X_n) &= \langle b^n \rangle \subseteq \pi_1(X_n). \end{aligned}$$

Then by **Van Kampen 2.4.3**,

$$\pi_1(\mathbf{R}^3 \setminus K) \cong \pi_1(S^3 \setminus K) \cong \pi_1(X) \cong {\langle a \rangle * \langle b \rangle} \big/ {\langle\langle a^m b^{-n} \rangle\rangle}.$$

We make a quick aside for algebra.

Definition 2.4.7. Let S be a generating set. Any group G can be expressed as a free group on the generating set $F(S)$, modulo some relations $R \subseteq F(S)$, i.e., $G = {F(S)} \big/ {\langle\langle R \rangle\rangle}$. We denote the **group presentation of** G to be $\langle S \mid R \rangle$.

In other words, the fundamental group of the knot complement in **Example 2.4.6** is $\langle a, b \mid a^m b^{-n} \rangle = \langle a, b \mid a^m = b^n \rangle$.

Theorem 2.4.8. *Let X be a CW-complex and let $x_0 \in X^2$. Then $\pi_1(X, x_0) \cong \pi_1(X^2, x_0)$. In other words, the fundamental group doesn't see above the 2-skeleton.*

Proof. We show it for when X has dimension 3. Let $X = X^2 \sqcup_\varphi D^3$, where $\varphi : \partial D^3 \to X^2$ is an attaching map.

Fix a point $p \in \mathrm{Int}(D^3)$ so that $D^3 \setminus \{p\}$ deformation retracts onto ∂D^3 which is homeomorphic to S^2. Then $X = X \setminus \{p\} \cup e^3$. $X \setminus \{p\}$ deformation retracts onto X^2, and the fundamental group of e^3 is trivial. We just check the intersection: $X \setminus \{p\} \cap e^3$ is homeomorphic to $S^2 \times (-1, 1)$, which is simply connected. Therefore, by **Van Kampen 2.4.3**, we see that

$$\pi_1(X) \cong \pi_1(X^2) * \pi_1(e^3) \cong \pi_1(X^2),$$

as claimed. □

Theorem 2.4.9. *If we attach a 2-cell via the attaching map $\varphi : \partial D^2 \cong S^1 \to X$, then fix an $x_0 \in X$, fix a point $s_0 \in \varphi(\partial D^2)$, and fix a path γ from x_0 to s_0. Then we get a loop $\gamma\varphi\overline{\gamma}$ based at x_0, and so $[\gamma\varphi\overline{\gamma}] \in \pi_1(X, x_0)$.*

Then

$$\pi_1(X \sqcup_\varphi D^2) = {}^{\pi_1(X)}\!\big/_{\langle\langle \gamma\varphi\overline{\gamma} \rangle\rangle}.$$

Proof. Similar to **Theorem 2.4.8**, we have $Y = X \sqcup_\varphi D^2 = Y \setminus \{p\} \cup e^2$, where $Y \setminus \{p\}$ deformation retracts onto X and $\pi_1(e^2) = 0$. See that $Y \setminus \{p\} \cap e^2 \cong S^1 \times (-1, 1)$, an annulus. By **Van Kampen 2.4.3**, we have

$$\begin{aligned}\pi_1(Y) &= {}^{\pi_1(X) * \pi_1(e^2)}\!\big/_{\langle\langle \iota_x(\alpha)\iota_{e^2}(\alpha)^{-1} \mid \alpha \in \pi_1(Y \setminus \{p\} \cap e^2) \rangle\rangle} \\ &= {}^{\pi_1(X)}\!\big/_{\langle\langle \iota_x(\alpha)\iota_{e^2}(\alpha)^{-1} \mid \alpha \in \mathbf{Z} \rangle\rangle}.\end{aligned}$$

To find explicitly the generator α of $\pi_1(Y \setminus \{p\} \cap e^2)$, we have $\iota_x(\alpha) = \gamma\varphi\overline{\gamma}$ and $\iota_{e^2}(\alpha) = 1$, so ultimately, we have that

$$\pi_1(X \sqcup_\varphi D^2) = {}^{\pi_1(X)}\!\big/_{\langle\langle \gamma\varphi\overline{\gamma} \rangle\rangle},$$

as desired. □

Example 2.4.10. Let $X = S_g$, the orientable surface of genus g. Let's recall the cell structure, which we described in **Example 2.1.6**:

The 0-skeleton is a single 0-cell.

The 1-skeleton is $X^1 \bigvee_{i=1}^{2g} {S^1}_i$.

Then $S_g = X^1 \sqcup_\varphi D^2$, where $\varphi(\partial D^2) = a_1 b_1 {a_1}^{-1} {b_1}^{-1} ... a_g b_g {a_g}^{-1} {b_g}^{-1}$.

Therefore, $\pi_1(S_g) = \langle a_1, b_1, ..., a_g b_g \mid [a_1, b_1]...[a_g, b_g] \rangle$.

Corollary 2.4.11. *Surfaces S_g and S_h are homotopy equivalent if and only if $g = h$.*

Proof. Consider the abelianization:

$$(\pi_1(S_g))_{ab} \cong (\langle a_1, b_1, ..., a_g b_g \mid [a_1, b_1]...[a_g, b_g] \rangle)_{ab} \cong \mathbf{Z}^{2g}.$$

Since the fundamental group factors through the abelianization[12], we have the commutative diagram:

[12] I may have these wrong, but the diagram is correct.

$$\begin{tikzcd} \mathbf{F}_{2g} \arrow[rr] \arrow[dr] & & \mathbf{Z}^{2g} \\ & \pi_1(S_g) \arrow[ur] & \end{tikzcd}$$

We see that two surfaces have the same fundamental group if and only if $g = h$. □

Theorem 2.4.12. *Every group G is the fundamental group of a topological space.*

Proof. Let $G = \langle \{s_\alpha\} \mid \{r_\beta\} \rangle$. Then construct a CW-complex where X^0 is a single 0-cell, $X^1 = \bigvee_\alpha S^1{}_\alpha$, and X^2 is constructed by attaching 2-cells $e^2{}_\beta$ so that $r_\beta = s_{\beta_1}...s_{\beta_n}$ describes an attaching map φ_β. Then by **Theorem 2.4.9**,

$$\pi_1(X) = {}^{F(\{s_\alpha\})}\!/\!_{\langle\langle\{r_\beta\}\rangle\rangle} = G.$$

□

2.5 Section 5: Theory of Covering Spaces

Definitions: degree of cover, semilocally simply connected, isomorphism of covers, universal cover
Main Idea: We discuss covering spaces and their sheets. We show existence and uniqueness of lifts. We define the univeral cover, which exists if X is path connected, locally path connected, and semilocally simply connected.

Recall the following definitions; most are pulled from **Sections 53** and **54**:

- A covering space **Definition 1.53.1**: a map $p : \widetilde{X} \to X$ where $p^{-1}(U) = \coprod U_\alpha$ with $p|_{U_\alpha} : U_\alpha \to U$ a homeomorphism.
- A lift **Definition 1.54.1**: given a covering space, if $f : Y \to X$, then a lift is $\widetilde{f} : Y \to \widetilde{X}$ such that $f = p\widetilde{f}$.
- The **Homotopy Lifting Property 1.54.3**: given a covering space, a homotopy $f_t : Y \to X$, and a lifting $\widetilde{f_0} : Y \to \widetilde{X}$, there exists a unique lift $\widetilde{f_t} : Y \to \widetilde{X}$.

  We remark the following corollary of the **HLP 1.54.3**:
  For all homotopies of paths $f_t : I \to X$ and $\widetilde{x_0} \in p^{-1}(f_t(0))$, there exists a unique homotopy of paths $\widetilde{f_t} : I \to \widetilde{X}$ which is a lift and $\widetilde{f_t}(0) = \widetilde{x_0}$. Note that $f_t(0)$ is independent of t, since a homotopy of paths fixes the endpoints.

- The **Path Lifting Property 1.54.2**: for all paths $f : I \to X$ and choice of base point $\widetilde{x_0} \in p^{-1}(f(0))$, there is a unique lift $\widetilde{f} : I \to \widetilde{X}$ such that $\widetilde{f}(0) = \widetilde{x_0}$.

- A lifting correspondence **Definition 1.54.5**: given a covering space, $\varphi : \pi_1(X, x_0) \to p^{-1}(x_0)$. Fix $\widetilde{x_0} \in p^{-1}(x_0)$; then $\varphi([\gamma]) = \widetilde{\gamma}(1)$ where $p\widetilde{\gamma} = \gamma$ (i.e. $\widetilde{\gamma}$ is a lift of γ) and $\widetilde{\gamma}(0) = \widetilde{x_0}$.

- Recall also that if $\widetilde{X}$ is path connected, the φ is surjective, and if $\widetilde{X}$ is simply connected, then φ is injective. (This is just **Theorem 1.54.7**.)

Theorem 2.5.1. *The map $p_* : \pi_1(\widetilde{X}, \widetilde{x_0}) \to \pi_1(X, x_0)$ is injective. The image is $\{[\gamma] \mid \varphi([\gamma]) = \widetilde{x_0}\}$, which is the set of paths that lift to loops in the cover $\widetilde{X}$.*

Proof. We show that $\ker p_* = \{1\}$. If $p_*[\widetilde{\gamma}]$ is trivial, then there exists a nulhomotopy of $\gamma = p\widetilde{\gamma}$. Call it $\gamma_t : I \to X$. The **HLP 1.54.3** implies that there exists $\widetilde{\gamma}_t : I \to \widetilde{X}$ which is a nulhomotopy of $\widetilde{\gamma}$. Therefore $\ker p_* = \{1\}$, as desired, and p_* is injective.

Now to show that the image is as claimed, see that $\varphi([\gamma]) = \widetilde{x_0}$ if and only if $\gamma = p\widetilde{\gamma}$ where $\widetilde{\gamma}(0) = \widetilde{x_0} = \widetilde{\gamma}(1)$. This is the case if and only if $[\gamma] = p_*[\widetilde{\gamma}]$ for some $[\widetilde{\gamma}] \in \pi_1(\widetilde{X}, \widetilde{x_0})$. $\square$

Example 2.5.2. Let $p : S^1 \to S^1$ by $z \mapsto z^n$. We know that $\pi(S^1) \cong \mathbf{Z} \cong \langle a \rangle$, so the image of p_* is $\langle a^n \rangle$.

Definition 2.5.3. If $p : \widetilde{X} \to X$ is a covering space and X is connected, then we define the **degree of the cover** (also called the **number of sheets**) to be the cardinality of $p^{-1}(x)$ for any $x \in X$.

Proposition 2.5.4. *As X is connected, the degree of the cover doesn't depend on x.*

Example 2.5.5. If $p : S^1 \to S^1$, $p(z) = z^n$, then the degree of the cover is $|n|$.

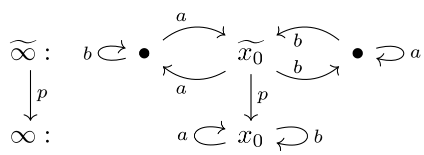
Lemma 2.5.6. *The number of sheets of $p : \widetilde{X} \to X$ when $\widetilde{X}$ and X are path connected is the index $[\pi_1(X, x_0) : p_*\pi_1(\widetilde{X}, \widetilde{x_0})]$.*

Proof. Fix a loop $g : I \to X$ in X based at x_0. Then $\widetilde{g} : I \to \widetilde{X}$ is a path starting at $\widetilde{x_0}$. Then for $[h] \in p_*\pi_1(\widetilde{X}, \widetilde{x_0})$, hg has the lift $\widetilde{hg}$ which ends where $\widetilde{g}$ does, since $\widetilde{h}$ is a loop. Then there is a well-defined function $\varphi : p_*\pi_1(\widetilde{X}, \widetilde{x_0})[g] \to p^{-1}(x_0)$ by $\pi_1(\widetilde{X}, \widetilde{x_0})[g] \mapsto \widetilde{g}(1)$.

Since $\widetilde{X}$ is path connected, there exists a path $\widetilde{g}$ from any element in $p^{-1}(x_0)$ to $\widetilde{x_0}$ such that g is a loop based at x_0 in X. Thus φ is surjective. If we can see that φ is injective, then a bijection does it, showing there are exactly as many sheets as the index.

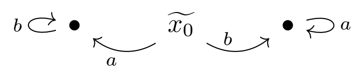
For injectiveness, if $\varphi(p_*\pi_1(\widetilde{X}, \widetilde{x_0})[g_1]) = \varphi(p_*\pi_1(\widetilde{X}, \widetilde{x_0})[g_2])$, then $g_1\overline{g_2}$ lifts to a loop in $\widetilde{X}$ based at $\widetilde{x_0}$, so $[g_1][\overline{g_2}] \in p_*\pi_1(\widetilde{X}, \widetilde{x_0})$, so $p_*\pi_1(\widetilde{X}, \widetilde{x_0})[g_1] = p_*\pi_1(\widetilde{X}, \widetilde{x_0})[g_2]$. $\square$

Example 2.5.7. Consider the covering space pictured:



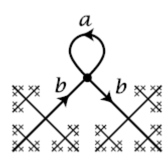
Note that we are being lazy with notating paths in the cover.

Then there are three sheets, since being on e.g. loop a in the base space lifts to three loops $\widetilde{a}$ in $\widetilde{\infty}$. We can also list the loops in $\widetilde{\infty}$ which generate all loops in $\widetilde{\infty}$. They are $\langle a^2, b^2, aba^{-1}, bab^{-1}\rangle$. To see this, collapse the maximal spanning tree of $\widetilde{\infty}$. In this case the maximal tree is



which evidently give aba^{-1} and bab^{-1} (a^2 and b^2 were clear). Then any loop, for instance bab, is a product of generators. See that $bab = bab^{-1}b^2$.

Example 2.5.8. Consider the same base space ∞, with the cover now



This is an infinite-sheeted cover! The only loop in the cover is a, so we have $\langle a\rangle$, a subgroup of $\langle a, b\rangle$.

Theorem 2.5.9 (**Lifting Criterion**). *Let $p : \widetilde{X} \to X$ be a covering space. Let $f : Y \to X$. Suppose Y is path connected and locally path connected. Then a lift $\widetilde{f} : Y \to \widetilde{X}$ of f exists if and only if $f_*\pi_1(Y, y_0) \subseteq p_*\pi_1(\widetilde{X}, \widetilde{x_0})$.*

Proof. First, suppose $\widetilde{f}$ exists. Then $f = p\widetilde{f}$, so $f_* = p_*\widetilde{f}_*$. Then $f_*\pi_1(Y, y_0) = p_*\widetilde{f}_*\pi_1(Y, y_0) \subseteq p_*\pi_1(\widetilde{X}, \widetilde{x_0})$.

Now, suppose $f_*\pi_1(Y, y_0) \subseteq p_*\pi_1(\widetilde{X}, \widetilde{x_0})$. Let $y \in Y$ and fix a path $\gamma : I \to Y$ from y_0 to y. Then, applying f, we have a path $f\gamma : I \to X$ from x_0 to $f(y)$.

The path $f\gamma$ has the unique lifting property, by **PLP 1.54.2**. Let $\widetilde{f_\gamma} : I \to \widetilde{X}$ be the lift with $\widetilde{f_\gamma}(0) = \widetilde{x_0}$. Set $\widetilde{f}(y) = \widetilde{f_\gamma}(1)$.

> We briefly remark that this is well-defined; i.e., independent of choice of γ. Let $\gamma' : I \to Y$ from y_0 to y be another path. Then the loop $[h] = [f\gamma'\overline{f\gamma}] = f_*[\gamma'\gamma] \in f_*\pi_1(Y, y_0) \subseteq p_*\pi_1(\widetilde{X}, \widetilde{x_0})$. Thus h is the image of a loop $\widetilde{h}$ in $\widetilde{X}$ based at $\widetilde{x_0}$. By the uniqueness of lifting paths, the first half of $\widetilde{h}$ is $\widetilde{f_{\gamma'}}$, and the second half of $\widetilde{h}$ is $\overline{\widetilde{f_\gamma}}$. Thus, the midpoint of $\widetilde{h}$ is $\widetilde{f_{\gamma'}}(1) = \widetilde{f_\gamma}(1)$.

It remains to be seen that the lift $\widetilde{f}$ is continuous. Let $U \subseteq X$ be an open neighborhood of $f(y)$ such that there is a lift $\widetilde{U} \subseteq \widetilde{X}$ where $\widetilde{f}(y) \in \widetilde{U}$ and $p : \widetilde{U} \to U$ is a homeomorphism. Then let V be a neighborhood of y such that $f(V) \subseteq U$. For all $y' \in V$, as Y is path connected and locally path connected, there exists a path η from y to y'. Then $\gamma\eta$ is a path from y_0 to y'. Therefore $f\gamma f\eta$ lifts to $\widetilde{f\gamma f\eta}$ where $\widetilde{f\eta} = p^{-1}f\eta$. Thus, $\widetilde{f}(V) \subseteq \widetilde{U}$, and $\widetilde{f}\Big|_V = p^{-1}f$, so $\widetilde{f}$ is continuous. $\square$

Theorem 2.5.10. *Suppose Y is connected. If $\widetilde{f_1}$ and $\widetilde{f_2}$ are lifts of f and $\widetilde{f_1}(y_0) = \widetilde{f_2}(y_0)$ for some $y_0 \in Y$, then $\widetilde{f_1} = \widetilde{f_2}$.*

The **Lifting Criterion 2.5.9** characterizes existence of lifts. This theorem characterizes uniqueness; if two lifts of a function agree on a point, then they agree entirely.

Proof. For $y \in Y$, let $U \subseteq X$ be an open set such that $f(y) \in U$ and $p^{-1}(U) = \coprod_\alpha U_\alpha$ with $p|_{U_\alpha} : U_\alpha \to U$ is a homeomorphism. Then suppose $\widetilde{f_1}(y) \in U_{\alpha_1}$ and $\widetilde{f_2}(y) \in U_{\alpha_2}$. By continuity, fix a neighborhood $V \subseteq Y$ of y such that $\widetilde{f_1}(V) \subseteq U_{\alpha_1}$ and $\widetilde{f_2}(V) \subseteq U_{\alpha_2}$. There are two cases:

1. If $\widetilde{f_1}(y) = \widetilde{f_2}(y)$, then $U_{\alpha_1} = U_{\alpha_2}$. Thus $\widetilde{f_1} = \widetilde{f_2}$ on V, as p is injective on U_α and $p\widetilde{f_1} = f = p\widetilde{f_2}$. We conclude that the set $\{x \in Y \mid \widetilde{f_1}(x) = \widetilde{f_2}(x)\}$ is open.

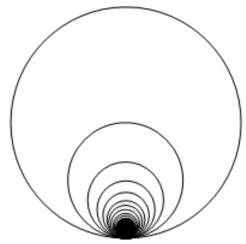
2. If $\widetilde{f_1}(y) \neq \widetilde{f_2}(y)$, then $U_{\alpha_1} \neq U_{\alpha_2}$. Thus $\widetilde{f_1} \neq \widetilde{f_2}$ on V. We conclude that the set $\{x \in Y \mid \widetilde{f_1}(x) \neq \widetilde{f_2}(x)\}$ is open. Equivalently, the set $\{x \in Y \mid \widetilde{f_1}(x) = \widetilde{f_2}(x)\}$ is closed.

So $\{x \in Y \mid \widetilde{f_1}(x) = \widetilde{f_2}(x)\}$ is open and closed in Y. As Y is connected, by **Definition 1.23.2**, $\{x \in Y \mid \widetilde{f_1}(x) = \widetilde{f_2}(x)\}$ clopen implies it is either Y or $\emptyset$. But by hypothesis, $\widetilde{f_1}(y_0) = \widetilde{f_2}(y_0)$, so $y_0 \in \{x \in Y \mid \widetilde{f_1}(x) = \widetilde{f_2}(x)\}$. Thus $\{x \in Y \mid \widetilde{f_1}(x) = \widetilde{f_2}(x)\} = Y$, so $\widetilde{f_1} = \widetilde{f_2}$. $\square$

Definition 2.5.11. A space X is **semilocally simply connected** if for all $x \in X$, there exists a neighborhood $U \subseteq X$ of x such that the map induced by inclusion $\iota_* : \pi_1(U, x) \to \pi_1(X, x)$ is trivial; i.e. $\iota_*([f]) = 1$.

Example 2.5.12. CW-complexes are semilocally simply connected.

Example 2.5.13. The Hawaiian earring, pictured below, is not semilocally simply connected. Any neighborhood of the intersection point contains infinitely many circles, so $\pi_1(U, x) \cong \pi_1(X, x)$.



Our goal now is to figure out what conditions we must impose on X to get a simply connected cover $\widetilde{X}$ (e.g., $\mathbf{R} \xrightarrow{p} S^1$).

Lemma 2.5.14. *Let $\widetilde{X} = \{[\gamma] \mid \gamma : I \to X, \gamma(0) = x_0\}$, and let $p : \widetilde{X} \to X$ be defined by $p([\gamma]) = \gamma(1)$. Then the set $\mathcal{B}$ consisting of path connected open sets $U \subseteq X$ such that $\pi_1(U) \to \pi_1(X)$ is trivial is a basis for the topology on X.*

Proof. Given a neighborhood $V \subseteq X$ of x and $U \in \mathcal{B}$ such that $x \in U$, then $x \in V \cap U$ so we have

$$\begin{tikzcd} \pi_1(V \cap U, x) \arrow[rr] \arrow[dr] & & \pi_1(X, x) \\ & \pi_1(U, x) \arrow[ur, "0"] & \end{tikzcd}$$

Since $\pi_1(U) \to \pi_1(X)$ is trivial, so too must be $\pi_1(V \cap U) \to \pi_1(X)$. Therefore $V \cap U \in \mathcal{B}$, and $\mathcal{B}$ is a basis for the topology on X. $\square$

Lemma 2.5.15. *Let $U \in \mathcal{B}$ and let $\gamma : I \to X$ be a path where $\gamma(0) = x_0$ and $\gamma(1) \in U$. We define $U_{[\gamma]} = \{[\gamma\eta] \mid \eta : I \to U, \gamma(1) = \eta(0)\} \subseteq \widetilde{X}$. Then $\{U_{[\gamma]}\} = \widetilde{\mathcal{B}}$ is a basis for the topology on $\widetilde{X}$.*

Proof. We first claim that if $[\gamma'] \in U_{[\gamma]}$, then $U_{[\gamma]} = U_{[\gamma']}$. In other words, in the "stack of pancakes" of a covering space, if two pancakes intersect, then they are the same pancake.

> Let $[\gamma'] \in U_{[\gamma]}$. Then we can write $[\gamma'] = [\gamma\eta]$ for some η. Then $[\gamma'\eta'] = [\gamma\eta\eta']$, so $U_{[\gamma']} \subseteq U_{[\gamma]}$.
>
> On the other hand, $[\gamma\eta'] = [\gamma\eta\overline{\eta}\eta'] = [\gamma'\overline{\eta}\eta']$, so $U_{[\gamma]} \subseteq U_{[\gamma']}$. Therefore, $U_{[\gamma]} = U_{[\gamma']}$.

Now, we show that $\{U_{[\gamma]}\}$ is a basis for $\widetilde{X}$. Let $[\gamma''] \in U_{[\gamma]} \cap V_{[\gamma']}$ where $U_{[\gamma]}, V_{[\gamma']} \in \mathcal{B}$. Then $U_{[\gamma'']} = U_{[\gamma]}$ and $V_{[\gamma'']} = V_{[\gamma]}$. Thus if $W \in \mathcal{B}$ such that $\gamma''(1) \in W \subseteq U \cap V$, since $\mathcal{B}$ is a basis on X, then we have $[\gamma''] \in W_{[\gamma'']} \subseteq U_{[\gamma'']} \cap V_{[\gamma'']} = U_{[\gamma]} \cap V_{[\gamma]}$. Therefore, $\widetilde{\mathcal{B}}$ is a basis for a topology on $\widetilde{X}$. $\square$

Lemma 2.5.16. *The map $p : U_{[\gamma]} \to U$ is a homeomorphism.*

Proof. First, we claim that $p : U_{[\gamma]} \to U$ is a bijection.

To see surjectivity, U is path connected, **Theorem 1.54.7** gives us p surjective.

To see injectivity, by assumption the inclusion map $\pi_1(U) \to \pi_1(X)$ is trivial, so if $p([\gamma\eta]) = p([\gamma\eta'])$, then $\eta\overline{\eta'}$ is a loop in U, so $\gamma\eta \simeq \gamma\eta'$, and p is injective.

Now since p is a bijection, p induces a bijection between subsets $V_{[\gamma']} \subseteq U_{[\gamma]}$ and $V \subseteq U$. Thus p is a homeomorphism. $\square$

From this we can conclude that the notation is appropriate; $p : \widetilde{X} \to X$ is a covering space.

Lemma 2.5.17. *Let X be path connected, locally path connected, and semilocally simply connected. Let $\widetilde{X}, p$ be as described above. Let $[x_0]$, a constant path in $\widetilde{X}$, be the base point of $\widetilde{X}$. Then $\widetilde{X}$ is path connected.*

Proof. Given $[\gamma] \in \widetilde{X}$, define a homotopy $\gamma_t : I \to X$ by $\gamma_t(s) = \gamma(ts)$. Then $\gamma_1 = \gamma$ and $\gamma_0 = [x_0]$. Thus there is a path from any $[\gamma]$ to $[x_0]$ to any $[\gamma']$. $\square$

Theorem 2.5.18. *If X is path connected, locally path connected, and semilocally simply connected, then $\widetilde{X}$ is simply connected.*

Proof. Let $f : I \to \widetilde{X}$ be a loop based at $[x_0]$; we show it is nulhomotopic.

Define the homotopy $\gamma_t(s) = \gamma(ts)$. Then $[\gamma_t]$ is a lift of the path γ to $\widetilde{X}$. Since $[\gamma_0] = [x_0] = f(0)$, the uniqueness of lifts implies that $[\gamma_t] = f(t)$. Then $[\gamma_1] = f(1) = [x_0]$. But since $[\gamma_1] = [\gamma]$, γ is nulhomoptic. And since p is a cover, by **Theorem 2.5.1**, p_* is injective, so f is also nulhomotopic, as desired. $\square$

Thus **Theorem 2.5.18** gives sufficient conditions for $\widetilde{X}$ to be simply connected.

Definition 2.5.19. An **isomorphism of covers** is a homeomorphism $f : \widetilde{X_1} \to \widetilde{X_2}$ such that if $p_1 : \widetilde{X_1} \to X$ and $p_2 : \widetilde{X_2} \to X$ are covering spaces, then $p_1 = p_2 f$. In other words, the diagram commutes:

$$\begin{tikzcd} \widetilde{X_1} \arrow[rr, "f"] \arrow[dr, "p_1"'] & & \widetilde{X_2} \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

Theorem 2.5.20. *If X is path connected, locally path connected, and semilocally simply connected, then there exists a bijection between*

- *subgroups of $\pi_1(X, x_0)$ and covers $p : (\widetilde{X}, \widetilde{x_0}) \to (X, x_0)$ modulo base point preserving isomorphisms, and*
- *conjugacy classes of subgroups of $\pi_1(X, x_0)$ and covers $p : (\widetilde{X}, \widetilde{x_0}) \to (X, x_0)$ modulo isomorphism.*

Proof. Let $p : \widetilde{X} \to X$ be a simply connected covering space of X. Then fix a subgroup $H \leq \pi_1(X, x_0)$. Define an equivalence relation on $\widetilde{X}$ by $[\gamma] \sim [\gamma']$ if $\gamma(1) = \gamma'(1)$ and $[\gamma'\overline{\gamma}] \in H$. Let $X_H = {}^{\widetilde{X}}\!\big/_{\sim}$.

If $[\gamma] \sim [\gamma']$, then $\gamma(1) = \gamma'(1)$ and so $p([\gamma]) = p([\gamma'])$, and hence p factors through X_H. Further, if $[\gamma] \sim [\gamma']$, then $[\gamma\eta] \sim [\gamma'\eta]$, so the equivalence relation identifies $U_{[\gamma]}$ and $U_{[\gamma']}$. Thus, $p^H : X_H \to X$ is still a covering space.

Let $\widetilde{x_0}$ be the equivalence class of $[x_0] \in \widetilde{X}$. Then ${p^H}_* \pi_1(X_H, \widetilde{x_0}) = H$, since a loop γ in X lifts to a path from $[x_0]$ to $[\gamma]$ in $\widetilde{X}$. This path is a loop in X_H if and only if $[\gamma] \sim [x_0]$, which is exactly when $[\gamma] \in H$.

Now we prove the theorem. First, if $f : \widetilde{X_1} \to \widetilde{X_2}$ is an isomorphism preserving base points (i.e., $f(\widetilde{x_1}) = x_2$ for $f(\widetilde{x_1}) \in {p_1}^{-1}(x_0)$ and $x_2 \in {p_2}^{-1}(x_0)$), then $p_1 = p_2 f$ implies that $p_{1*}\pi_1(\widetilde{X_1}, \widetilde{x_1}) \subseteq p_{2*}\pi_1(\widetilde{X_2}, \widetilde{x_2})$. Since f is a homeomorphism, we can do the same argument to $p_2 = p_1 f^{-1}$, and thus $p_{1*}\pi_1(\widetilde{X_1}, \widetilde{x_1}) = p_{2*}\pi_1(\widetilde{X_2}, \widetilde{x_2})$.

For the other direction, assume $p_{1*}\pi_1(\widetilde{X_1}, \widetilde{x_1}) = p_{2*}\pi_1(\widetilde{X_2}, \widetilde{x_2})$. Then we may lift p_1 and p_2 to $\widetilde{p_1}$ and $\widetilde{p_2}$; see the commutative diagrams:

$$\begin{tikzcd}
 & (\widetilde{X_2}, \widetilde{x_2}) \arrow[d, "p_2"] \\
(\widetilde{X_1}, \widetilde{x_1}) \arrow[ur, dotted, bend left, "\widetilde{p_1}"] \arrow[r, "p_1"] & (X, x_0)
\end{tikzcd}
\qquad
\begin{tikzcd}
 & (\widetilde{X_1}, \widetilde{x_1}) \arrow[d, "p_1"] \\
(\widetilde{X_2}, \widetilde{x_2}) \arrow[ur, dotted, bend left, "\widetilde{p_2}"] \arrow[r, "p_2"] & (X, x_0)
\end{tikzcd}$$

We can then conjoin the diagrams, gluing p_2 to p_2, to get:

$$\begin{tikzcd}
 & & \widetilde{X_1} \arrow[dd, "p_1"] \\
 & \widetilde{X_2} \arrow[ur, dotted, bend left, "\widetilde{p_2}"] \arrow[dr, "p_2"] & \\
\widetilde{X_1} \arrow[ur, dotted, bend left, "\widetilde{p_1}"] \arrow[rr, "p_1"] & & X_0
\end{tikzcd}$$

Thus we can see that $\widetilde{p_2}\widetilde{p_1}$ is a lift of p_1 to $\widetilde{X_1}$, and $\widetilde{p_2}\widetilde{p_1}(\widetilde{x_1}) = \widetilde{x_1}$. Further note that $id_{\widetilde{X_1}}$ is also a lift such that $id_{\widetilde{X_1}}(\widetilde{x_1}) = \widetilde{x_1}$. Since lifts are unique, $\widetilde{p_2}\widetilde{p_1} = id_{\widetilde{X_1}}$. Similarly, one can show that $\widetilde{p_1}\widetilde{p_2} = id_{\widetilde{X_2}}$. Thus $\widetilde{X_1}$ and $\widetilde{X_2}$ are homeomorphic.

We remark only briefly that changing the basepoint corresponds to conjugation: Let $g = [p\gamma] \in \pi_1(X, x_0)$ where p is the covering map and γ is a path in $\widetilde{X}$ from $\widetilde{x_0}$ to $\widetilde{x_1}$. Then $p_*\pi_1(\widetilde{X}, \widetilde{x_1}) = g^{-1} p_*\pi_1(\widetilde{X}, \widetilde{x_0}) g$. $\square$

Definition 2.5.21. A simply connected covering space $\widetilde{X}$ is called a **universal cover**. Such a cover is unique up to base point preserving isomorphism.

We remark that the universal cover factors through any other cover. If $p : \widetilde{X} \to X$ is the universal cover and $p_0 : \widetilde{X_0} \to X$ is an other cover, there exists $\hat{p_1}$ unique up to base points such that the following diagram commutes:

$$\begin{tikzcd}
\widetilde{X} \arrow[dd, "p"'] \arrow[dr, dotted, "\hat{p_1}"] & \\
& \widetilde{X_0} \arrow[dl, "p_0"] \\
X &
\end{tikzcd}$$

2.6 Section 6: Deck Transformations and Group Actions

Definitions: deck transformation, deck group, normal cover, group action, covering space action
Main Idea: This is a section to sit with for a while. Outside sources might be nice.

Definition 2.6.1. A **deck transformation of a cover** $p : \widetilde{X} \to X$ is a homeomorphism $f : \widetilde{X} \to \widetilde{X}$ such that $p = pf$.

Definition 2.6.2. The group with elements deck transformations f and operation function composition is called the **deck group of the cover**. We write $G(\widetilde{X})$ to mean the deck group of $\widetilde{X}$.

We remark that f is a lift of p; see

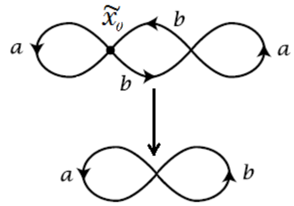
$$\begin{tikzcd}
& \widetilde{X} \arrow[d, "p"] \\
\widetilde{X} \arrow[ur, dotted, bend left, "f"] \arrow[r, "p"] & X
\end{tikzcd}$$

Thus if $\widetilde{X}$ is path connected, the unique lifting property implies that f is completely determined by where it sends a single point. This remark is very useful to see what the elements of the deck group are and determine the deck group itself.

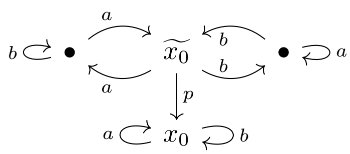
Example 2.6.3. $G(\mathbf{R} \to S^1) \cong \mathbf{Z}$. If $x \mapsto e^{2\pi ix}$, then $x + n \mapsto e^{2\pi i(x+n)} = e^{2\pi ix}$. Then we say S^1 acts on $\mathbf{R}$ by integer translations.

Example 2.6.4. $G(S^1 \xrightarrow{n} S^1) \cong \mathbf{Z}/n\mathbf{Z}$. This corresponds to rotations by multiples of $\frac{2\pi}{n}$.

Example 2.6.5. $G(\widetilde{X})$ where the cover is below is isomorphic to $\mathbf{Z}/2\mathbf{Z}$; simply swap vertex choice of $\widetilde{x_0}$.



Example 2.6.6. In $\widetilde{X}$ pictured below, $G(\widetilde{X}) \cong 0$ because since $\widetilde{x_0}$ must be fixed, it is the only vertex with two loops coming in and two loops going out.



Definition 2.6.7. A covering space $p : \widetilde{X} \to X$ is called **normal** or **regular** if for all $\widetilde{x_0}, \widetilde{x_1} \in p^{-1}(x_0)$, there exists a deck transformation $f : \widetilde{X} \to \widetilde{X}$ with $f \in G(\widetilde{X})$ such that $f(\widetilde{x_0}) = \widetilde{x_1}$; i.e., $G(\widetilde{X})$ transitively permutes $p^{-1}(x_0)$.

We remark that **Examples 2.6.3**, **2.6.4**, and **2.6.5** are normal covering spaces, but **Example 2.6.6** is not.

Theorem 2.6.8. *Let $p : (\widetilde{X}, \widetilde{x_0}) \to (X, x_0)$ be a cover where X and $\widetilde{X}$ are path connected and locally path connected. Let $H = p_*\pi_1(\widetilde{X}, \widetilde{x_0})$, the subgroup corresponding to the cover. Then*

1. *H is normal if and only if $p : \widetilde{X} \to X$ is normal, and*
2. *$G(\widetilde{X}) = {}^{N(H)}\!/\!_{H}$ where $N(H) = \{g \mid gHg^{-1} = H\}$ is the normalizer of H.*

Proof. We prove each claim in correspondence.

1. To see that H is normal if and only if $p : \widetilde{X} \to X$ is normal, let $\widetilde{x_0}, \widetilde{x_1} \in p^{-1}(x_0)$. Let $[\gamma] \in \pi_1(X, x_0)$ lift to a path from $\widetilde{x_0}$ to $\widetilde{x_1}$. Then see that $p_*\pi_1(\widetilde{X}, \widetilde{x_1}) = [\gamma]^{-1} p_*\pi_1(\widetilde{X}, \widetilde{x_0})[\gamma] = [\gamma]^{-1} p_* H[\gamma]$.

$$\begin{tikzcd} & (\widetilde{X}, \widetilde{x_0}) \arrow[d, "p"] \\ (\widetilde{X}, \widetilde{x_1}) \arrow[ur, dotted, bend left, "f"] \arrow[r, "p"] & (X, x_0) \end{tikzcd}$$

   Thus $[\gamma] \in N(H)$ if and only if $p_*\pi_1(\widetilde{X}, \widetilde{x_1}) = p_*\pi_1(\widetilde{X}, \widetilde{x_0})$, which is the case if and only if there exists a deck transformation taking $\widetilde{x_1}$ to $\widetilde{x_0}$, by the **Lifting Criterion 2.5.9**.

2. To see that $G(\widetilde{X}) = {}^{N(H)}\!/\!_{H}$, we define a homomorphism $N(H) \to G(\widetilde{X})$ by $\varphi([\gamma]) = f$, where f is the deck transformation sending $\widetilde{x_0}$ to $\widetilde{\gamma}(1)$. We

show that φ is a homomorphism, surjective, and then show that $\ker\varphi = H$, so $G(\widetilde{X}) \cong {}^{N(H)}\!/\!_{H}$.

To see homomorphism, let $[\gamma'] \in N(H)$. Let $\widetilde{\gamma}'$ be the lift starting at $\widetilde{x_0}$. Then the lift of $\gamma\gamma'$ is the path $\widetilde{\gamma} f(\widetilde{\gamma}')$. This path ends at $f(\widetilde{\gamma}'(1)) = ff'(\widetilde{x_0})$, where f' is the deck transformation for $[\gamma']$. Then see that $\varphi([\gamma\gamma']) = ff' = \varphi([\gamma])\varphi([\gamma'])$. Thus φ is a homomorphism.

To see surjection, if $f \in G(\widetilde{X})$, then just take the path $\widetilde{\gamma}$ from $\widetilde{x_0}$ to $f(\widetilde{x_0})$. Then $\varphi([p\widetilde{\gamma}]) = f$, as desired.

Finally, we show $\ker\varphi = H$, to complete the proof. Simply put, the kernel is the loops that lift to loops based at $\widetilde{x_0}$. This is exactly $p_*\pi_1(\widetilde{X}, \widetilde{x_0}) = H$. Thus $G(\widetilde{X}) = {}^{N(H)}\!/\!_{H}$, as desired. $\square$

Example 2.6.9. Recall **Example 2.6.3**. Since $\mathbf{R}$ is simply connected, $p_*\pi_1(\mathbf{R}) = 0$, and thus $G(\widetilde{X}) = \pi_1(S^1) = \mathbf{Z}$.

Example 2.6.10. Recall **Example 2.6.4**. Here $p_*\pi_1(S^1) = n\mathbf{Z}$, and $G(\widetilde{X}) = {}^{\mathbf{Z}}\!/\!_{n\mathbf{Z}}$.

Example 2.6.11. Here we see $p_*\pi_1(\widetilde{X}, \widetilde{x_0}) = \langle a, b^2, bab^{-1}\rangle$. Do the same maximal tree decomposition as **Example 2.5.7**.

Example 2.6.12. Recall **Example 2.6.6**. Recall that in **Example 2.5.7** we saw that $p_*\pi_1(\widetilde{X}, \widetilde{x_0}) = \langle a^2, b^2, aba^{-1}, bab^{-1}\rangle$.

Definition 2.6.13. A **group action** is a homomorphism $\rho : G \to \mathrm{Aut}(X)$, which we often write $\rho(g)(x) = gx$, such that

1. $e_G x = x$, and
2. $(gh)x = g(hx)$.

Definition 2.6.14. An action of a group G on a space Y is called a **covering space action** if for all $y \in Y$, there exists $U \subseteq Y$ a neighborhood of y such that if $g_1(U) \cap g_2(U) \neq \emptyset$, then $g_1 = g_2$. Equivalently, if $g(U) \cap U \neq \emptyset$, then $g = e_G$.

Example 2.6.15. $G(\widetilde{X})$, the deck group on the covering space, acting on $\widetilde{X}$ is a covering space action. Elements of $G(\widetilde{X})$ permute the stack of records. By the non-intersection of the records, if $g(U) \cap U \neq \emptyset$, then $g(\widetilde{x_0}) = \widetilde{x_0}$, so $g = id_{\widetilde{X}}$.

Proposition 2.6.16. *Suppose G acting on Y is a covering space action. Then*

1. *$p : Y \to {}^{Y}\!/\!_{G}$ is a normal covering space,*
2. *$G\left(Y \to {}^{Y}\!/\!_{G}\right) \cong G$ if Y is path connected, and*

3. $G \cong {}^{\pi_1\left({}^{Y}\!/_{G}\right)}\!\big/_{p_*\pi_1(Y)}$ *if Y is path connected and locally path connected.*

Corollary 2.6.17. *If Y is simply connected, then $p_*\pi_1(Y) = 0$, so by 3. in* ***Proposition 2.6.16****, $G \cong \pi_1\left({}^{Y}\!/_{G}\right)$.*

Example 2.6.18. $\mathbf{Z}$ acts on $\mathbf{R}$ by integer translations: $nx = x + n$. Given $x \in \mathbf{R}$, let $U = \left(x - \frac{1}{4}, x + \frac{1}{4}\right)$. Then translating U, i.e., $nU = \left(x + n - \frac{1}{4}, x + n + \frac{1}{4}\right)$, which intersects itself only when $n = 0$. Thus $nU \cap U \neq \emptyset$ only if $n = 0$.

By **Proposition 2.6.16**, $\pi_1\left({}^{\mathbf{R}}\!/_{\mathbf{Z}}\right) = \pi_1(S^1) = \mathbf{Z}$, as $\mathbf{R}$ is simply connected. (Recall **Example 2.6.3**.)

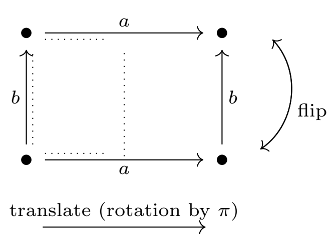
Example 2.6.19. Consider S^3 as $\{(z, w) \in \mathbf{C}^2 \mid |z|^2 + |w|^2 = 1\}$. Fix $p \geq 1$ and $0 < q \leq p$ such that $\gcd(p, q) = 1$. Then $\mathbf{Z}_p$ acts on S^3 by $n(z, w) = (e^{\frac{2\pi i n}{p}} z, e^{\frac{2\pi i n q}{p}} w)$. We check that this is a free action, which is weaker than a covering action:

If $z \neq 0$, then $z = e^{\frac{2\pi i n}{p}} z$, so $1 = e^{\frac{2\pi i n}{p}}$, so $n = 0 \mod p$; i.e., $n = 0 \in \mathbf{Z}_p$. If $w \neq 0$, then $w = e^{\frac{2\pi i n q}{p}} w$, so $1 = e^{\frac{2\pi i n q}{p}}$, and since $\gcd(p, q) = 1$, $n = 0 \in \mathbf{Z}_p$.

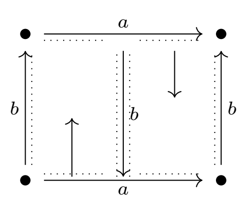
One of the above always happens in S^3, because if both $z = 0$ and $w = 0$, then $(z, w) \notin S^3$.

Such spaces; $L_{p,q} = {}^{S^3}\!/_{\mathbf{Z}_p}$, are called Lens spaces. By 3. in **Proposition 2.6.16**, $\pi_1(L_{p,q}) \cong \mathbf{Z}_p$.[13]

Example 2.6.20. $\mathbf{Z}_2$ acts on $S^1 \times S^1$ by the nontrivial element maps $(x, y) \mapsto (-x, \overline{y})$ where $-x$ is rotation by π and $\overline{y}$ is reflection. To see this, consider the polygonal representation of a torus:



The dotted region is called the fundamental domain, because every equivalence class, called an orbit, has a single representative here. To see this, we can overlay the fundamental domain over the torus twice with a flip; i.e.,



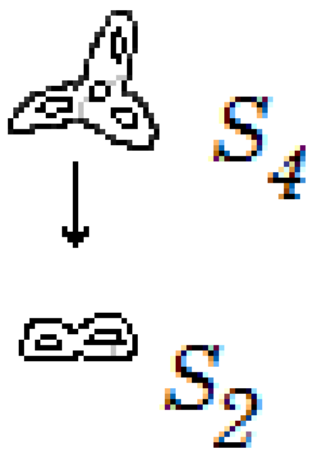
[13]We see more of Lens spaces in the summer; they motivate torsion, since the fundamental group (and other topological invariants we will define) doesn't see q. Torsion will!

From this picture we can clearly see that $S^1 \times S^1 / \mathbf{Z}_2$ is the Klein bottle, which we'll call K here. There are two Klein bottles in the cover of the torus in the picture above.

Thus $p : S^1 \times S^1 \to K$ is a covering map and $p_* \pi_1(S^1 \times S^1) \cong \mathbf{Z}^2$. We have

$$\mathbf{Z}_2 \cong \langle a^2, b \rangle \hookrightarrow \langle a, b \mid aba^{-1}b \rangle \xrightarrow{b \mapsto 1} \langle a \mid a^2 \rangle.$$

Example 2.6.21. $\mathbf{Z}_3$ acts by rotation by $\frac{2\pi}{3}$ on $S_4 \xrightarrow{p} S_2$. Then $S_4 / \mathbf{Z}_3$ is S_2 where we take a single arm of S_4 and cut it through the center circle, then gluing the gray cut line to form S_2. Thus $\pi_1(S_2)$ contains $\pi_1(S_4)$ as an index 3 subgroup.



This can clearly be generalized; if we have a surface S_g with d spokes, then there exists a d-sheeted cover $S_g \to S_{m+1}$ where m is the number of holes in a spoke, if $g = dm + 1$. Then $\mathbf{Z}_d$ acts on S_g by rotation by $\frac{2\pi}{d}$. We show later that this is if and only if.

2.7 Section 7: The Classification of Surfaces

Definitions: connect sum
Main Idea: Every surface is either the sphere, a connect sum of a bunch of tori, or a connect sum of a bunch of $\mathbf{RP}^2$.

Proposition 2.7.1. *Any compact connected surface is homeomorphic to the quotient space obtained from a collection of disjoint triangular regions by pasting edges together in pairs.*

Example 2.7.2. While we usually think of a torus as

$$\begin{tikzcd} \bullet \arrow[r, "a"] & \bullet \\ \bullet \arrow[u, "b"] \arrow[r, "a"] & \bullet \arrow[u, "b"] \end{tikzcd},$$

we can also consider the torus as

$$\begin{tikzcd}
\bullet \arrow[r, "a"] & \bullet \\
\bullet \arrow[u, "b"] \arrow[r, "a"'] \arrow[ur, "c"] & \bullet \arrow[u, "b"']
\end{tikzcd}.$$

Example 2.7.3. Similarly,

$$\begin{tikzcd}
\bullet \arrow[d, "b"] & \bullet \arrow[l, "a"] \\
\bullet \arrow[r, "a"] & \bullet \arrow[u, "b"']
\end{tikzcd} \quad \text{is the same surface as} \quad \begin{tikzcd}
\bullet \arrow[d, "b"] & \bullet \arrow[l, "a"] \\
\bullet \arrow[r, "a"] \arrow[ur, "c"] & \bullet \arrow[u, "b"']
\end{tikzcd}$$

Both surfaces are in fact $\mathbf{RP}^2$.

Corollary 2.7.4. *All polygons are triangulable, so a compact connected surface is the quotient of a polygon by identifying edges in pairs.*

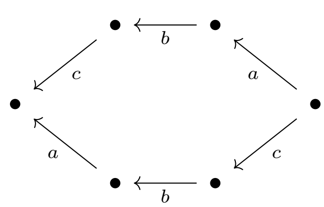
We remark that currently this is the more handy formulation, but when we regard surfaces as simplicial complexes, we actually need to decompose into triangles.

Corollary 2.7.4 allows us to have the following set-up for this section:

- Let P be an n-gon in the plane with a labeling scheme of edges, w, in ∂P.
- The scheme w is $w = (a_{i_1})^{\varepsilon_1}(a_{i_2})^{\varepsilon_2}...(a_{i_n})^{\varepsilon_n}$ where $\varepsilon_i \in \{-1, 1\}$. There are $\frac{n}{2}$ labels, since a label joins two edges of the n-gon.

Then $X = {}^{P}\!\diagup_{w}$ is the quotient space obtained by identifying edges via the labeling scheme w.

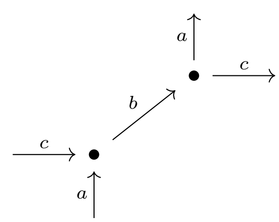
Example 2.7.5. Let $w = abca^{-1}b^{-1}c^{-1}$. Then we have the hexagon with labeling



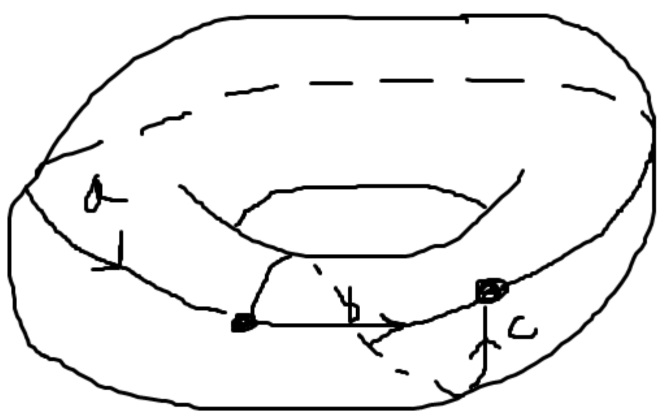
We claim this is a torus. To see this, consider the typical way we write the torus, which must be $w = aca^{-1}c^{-1}$. The single "corner" of this square torus thus sees the following:

$$\begin{tikzcd}
& {} & \\
{} \arrow[r, "c"] & \bullet \arrow[u, "a"] \arrow[r, "c"] & {} \\
& {} \arrow[u, "a"] &
\end{tikzcd}$$

We can see how this is the polygon with scheme $w = abca^{-1}b^{-1}c^{-1}$ by stretching the corner to be the side b:



We can see exactly the abc and the $a^{-1}b^{-1}c^{-1}$. We can also see how this becomes the donut-torus:



If you imagine contracting side b, you can see how this is exactly the usual way we make the torus.

Example 2.7.6. $w = abab^{-1}$ is the Klein bottle.

Example 2.7.7. $w = aa$ is $\mathbf{RP}^2$.

Example 2.7.8. $w = aa^{-1}$ is S^2.

Definition 2.7.9. If X and Y are compact surfaces with $A \subseteq X$ and $B \subseteq Y$ such that A and B are homeomorphic to the closed disk $D^2 \subseteq \mathbf{R}^2$, then the **connect sum of X and Y** is

$$X \# Y = X \setminus \mathrm{Int}(A) \bigsqcup_{\partial A = \partial B} Y \setminus \mathrm{Int}(B).$$

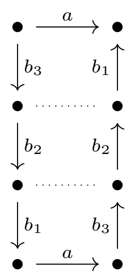
Example 2.7.10. If X is a compact surface, then $X \# S^2 = X$.

Example 2.7.11. $(S^1 \times S^1)\#(S^1 \times S^1) = S_2$.

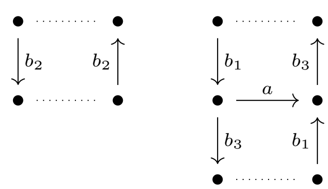
Example 2.7.12. $S_g = \underset{g}{\#}\, S^1 \times S^1$.

Example 2.7.13. $\mathbf{RP}^2 \# \mathbf{RP}^2$ is the Klein bottle.

To see this, write the Klein bottle as

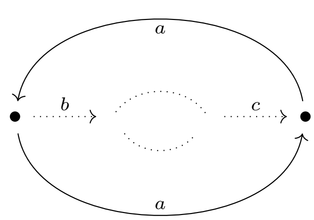


The dotted line is a circle in the Klein bottle. If we cut along this circle, we get two Möbius bands. See:

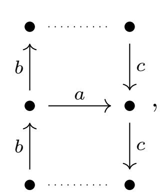


Therefore, the Klein bottle is two Möbius bands joined along their boundary. Thus, if we can show $\mathbf{RP}^2 \setminus \mathrm{Int}(D^2)$ is a Möbius band, that will prove the result.

To see this, consider



Cut along the dotted line and glue a to a. Then we have



which is a Möbius band, as desired.

Theorem 2.7.14 (**The Classification of Surfaces**). *If $g \geq 1$, then the orientable surface $S_g = \underset{g}{\#} S^1 \times S^1$ is the quotient $P/_w$ where the labeling scheme is $w = a_1 b_1 {a_1}^{-1} {b_1}^{-1} ... a_g b_g {a_g}^{-1} {b_g}^{-1}$, and the nonorientable surface $N_g = \underset{g}{\#} \boldsymbol{RP}^2$ is the quotient $P/_w$ where $w = a_1 a_1 a_2 a_2 ... a_g a_g$.*

This theorem classifies all the surfaces, as a surface is either orientable or not.

Proof. We begin by defining some elementary operations on schemes.

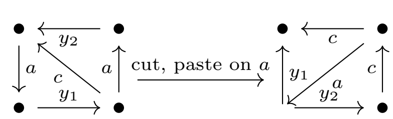
- We can cut: if $w = y_0 y_1$, then we can write $y_0 c$, $c^{-1} y_1$, where c does not appear anywhere else.
- We can paste: if we have two schemes $y_0 c$ and $c^{-1} y_1$, we can write $y_0 y_1$.
- We can relabel: we replace all occurences of a label with a new label that doesn't appear in the scheme.
- We can cyclically permute: if we have $y_0 y_1$, then we can write $y_1 y_0$.
- We can flip: if $w = (a_{i_1})^{\varepsilon_1} ... (a_{i_n})^{\varepsilon_n}$, then we can write $(a_{i_n})^{-\varepsilon_n} ... (a_{i_1})^{-\varepsilon_1}$.
- We can cancel: if $w = y_0 a a^{-1} y_1$, then we can write $y_0 y_1$ if a doesn't appear in y_0 or y_1.
- We can uncancel: if $w = y_0 y_1$ we can write $y_0 a a^{-1} y_1$ (note that we never use this).

We define two labeling schemes to be equivalent if one is obtained from another by a finite sequence of elementary operations.

We also call a labeling scheme proper if each label appears exactly twice, torus type if it is proper and each label has one positive exponent and one negative, and projective type if it is proper but not torus type. Thus, using this new language, we will show that every torus type scheme is equivalent to $a_1 b_1 {a_1}^{-1} {b_1}^{-1} ... a_g b_g {a_g}^{-1} {b_g}^{-1}$ and every projective type scheme is equivalent to $a_1 a_1 ... a_g a_g$, and the theorem will be proven.

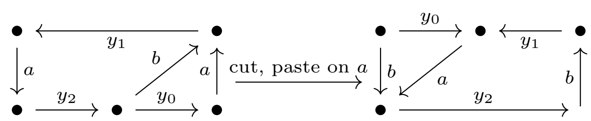
We first claim that $y_0 a y_1 a y_2$ is equivalent to $a a y_0 {y_1}^{-1} y_2$.

Suppose y_0 is empty. Then we have:



Thus we see we have $a y_1 a y_2$ is equivalent to $c^{-1} c^{-1} {y_1}^{-1} y_2$, which after relabeling, is $a a {y_1}^{-1} y_2$, as desired.

Now if y_1 or y_2 is empty, we can permute the labels to put the empty piece in the y_0 place. So it remains to show the case when y_0, y_1, y_2 are all nonempty. In this case, we have



> We thus have $by_1{y_0}^{-1}by_2$. Then $b(y_1{y_0}^{-1})by_2$, so we can apply the case when y_0 is empty to get $bb(y_0{y_1}^{-1})y_2$, as desired.

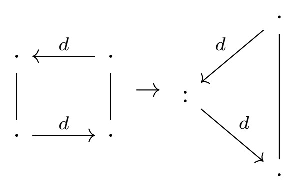
This claim shows that if w is projective type, then w is equivalent to a scheme of the same length of the form $a_1a_1a_2a_2...a_ka_kw_1$, where w_1 has torus type and $k \geq 1$. To see this is simple: find an appropriate label a, possible since w is projective type and use the claim to convert $y_0ay_1ay_2$ to $aay_0{y_1}^{-1}y_2$. If $y_0{y_1}^{-1}y_2$ has torus type, you are done. Otherwise, rewrite your scheme to be $aaz_0bz_1bz_2$. Then we have $(aaz_0)b(z_1)b(z_2)$, so again apply the claim to get $bbaaz_0{z_1}^{-1}z_2$. After finitely many steps, you are done.

Now we claim that if $w = w_0w_1$, where w_1 is torus type, and no two copies of the same label are adjacent, then w is equivalent to w_0w_2 of the same length as w_0w_1, where $w_2 = aba^{-1}b^{-1}w_3$, and w_3 is torus type or empty.

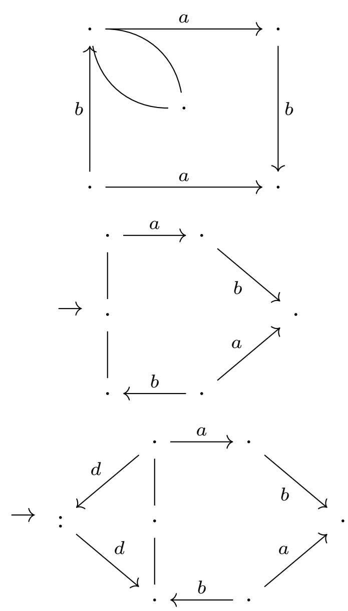
> We describe in words, but drawing the polygons out may prove to be helpful.
>
> Let $w = w_0y_1ay_2by_3a^{-1}y_4b^{-1}y_5$; this can always be chosen. Then make a cut c from the vertex where w_0 meets y_1 to the vertex where y_3 meets a. Then glue along a and relabel c as a.
>
> We now have $w = w_0ay_2by_3a^{-1}y_1y_4b^{-1}y_5$. Then make a cut c from the vertex where w_0 meets a to the vertex where a meets y_1. Then glue along b and relabel c as a and a as b.
>
> We now have $w = w_0ay_1y_4y_3b^{-1}a^{-1}by_2y_5$. Then make a cut c from the vertex where w_0 meets a to the vertex where a meets b. Then glue along b and relabel c as a and a as b^{-1}, and let $w_3 = y_1y_4y_3y_2y_5$.
>
> We get $w = w_0aba^{-1}b^{-1}w_3$, as desired.

We also claim that a scheme of the form $w_0ccaba^{-1}b^{-1}w_1$ is equivalent to $w_0aabbccw_1$.

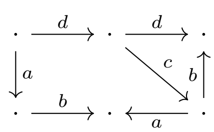
> The idea hinges on the fact that for a surface S, $S\#\mathbf{RP}^2\#(S^1 \times S^1)$ is homeomorphic to $S\#\mathbf{RP}^2\#\mathbf{RP}^2\#\mathbf{RP}^2$. We show this subclaim, and the claim follows.
>
> We know that $\mathbf{RP}^2\#\mathbf{RP}^2$ is the Klein bottle from **Example 2.7.13**, and that $\mathbf{RP}^2 \setminus D^2$ is a Möbius band. So to see what space is obtained by $\mathbf{RP}^2\#\mathbf{RP}^2\#\mathbf{RP}^2$ = Klein bottle$\#\mathbf{RP}^2$ is, we'll show that $[(S^1 \times S^1) \setminus D^2]$ glued to a Möbius band is homeomorphic to (Klein bottle)$\setminus D^2$ glued to a Möbius band. Here's our Möbius band:



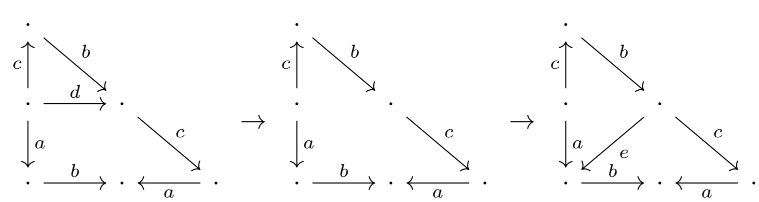
So, take the Klein bottle, remove a disk, and attach a Möbius band:



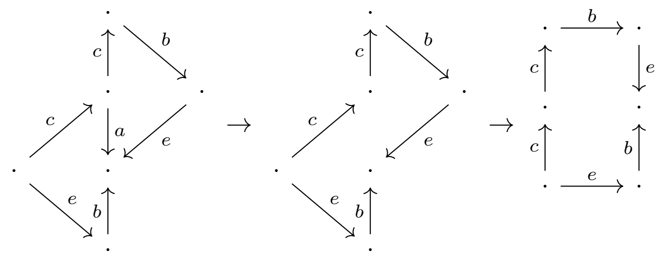
So we now have a hexagon with labeling scheme $w = ddb^{-1}ab^{-1}a^{-1}$. We wish to show that this labeling scheme represents $[(S^1 \times S^1) \setminus D^2]$ glued to a Möbius band. Indeed, see that:



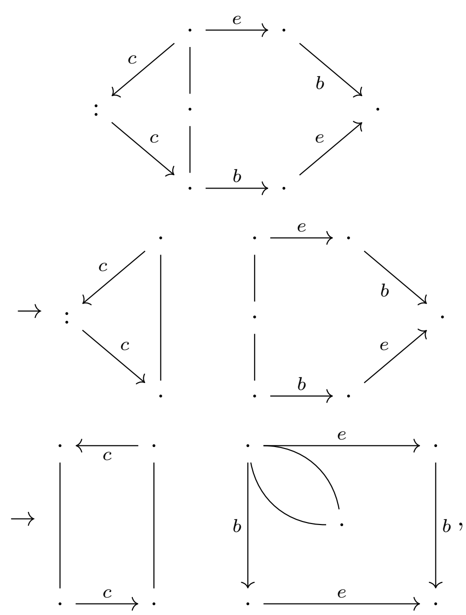
Cut along c; glue along d; identify a new side e:



Cut along e; glue along a; straighten out the polygon so it looks nicer:



Now, we wish to pull apart the labeling scheme $w = ccbeb^{-1}e^{-1}$ to show that it is a Möbius band stuck to a torus with a disk removed. In fact,



and, as desired, we have a Möbius band and a torus $ebe^{-1}b^{-1}$ with a disk removed.

Thus, we are finally ready to prove the theorem. Since it is several pages away by now, recall it to be that a compact surface is homeomorphic to S^2, S_g, or N_g for $g \geq 1$.

Let $X = {}^{P}\!/_{w}$. If the length of w is 2, we can enumerate all possibilities; we have either

- $w = aa$, and X is $\mathbf{RP}^2$, as we have seen in **Example 2.7.7**, or

- $w = aa^{-1}$, and X is S^2, as we have seen in **Example 2.7.8**.

If the length of w is 4, we again enumerate; we have:

- $w = aba^{-1}b^{-1}$, and $X = S^1 \times S^1 = S_1$,
- $w = abab^{-1}$, and X is the Klein bottle, which is equivalent to $aabb$, N_2,
- $w = abab$, so $w = cc$, and $X = \mathbf{RP}^2 = N_1$,
- $w = aa^{-1}bb^{-1}$, so $X = S^1$,
- $w = aabb^{-1}$ or $w = aa^{-1}bb$, which is also $X = N_1$, or
- $w = aba^{-1}b$, which is also the Klein bottle N_2.

If the length of w is 6, we no longer enumerate all possibilities; instead we use the claims we have proven to rewrite w. If w is of torus type, then we get $w = a_1b_1a_1{}^{-1}b_1{}^{-1}...a_gb_ga_g{}^{-1}b_g{}^{-1}$, and we have S_g. If w is of projective type, then $w = a_1a_1...a_ga_g$, and we have N_g. $\square$

Example 2.7.15. We can show that the Klein bottle $abab^{-1}$ is projective type as we claimed in the proof of the **Classification of Surfaces 2.7.14**, since we have two positive as. To see this, the sequence is

$$\begin{aligned} abab^{-1} &\rightsquigarrow abc, c^{-1}ab^{-1} && \text{cut} \\ &\rightsquigarrow bca, b^{-1}c^{-1}a && \text{permute} \\ &\rightsquigarrow bca, a^{-1}cb && \text{flip} \\ &\rightsquigarrow bccb && \text{paste} \\ &\rightsquigarrow bbcc && \text{permute} \end{aligned}$$

We remark for algebraists in the audience that compact surfaces form a monoid under connect sums. S^2 is the identity, and S_1 and N_1 are generators of infinite order. The monoid has the relation that $S_1\#N_1 = N_1\#N_1\#N_1$.

2.8 Section 8: Homology: Simplicial, Singular, Reduced, Relative, Cellular

Definitions: n-simplex, Δ-complex, chain group, boundary homomorphism, chain complex, homology groups, singular n-simplex, singular n-chain, singular chain group, singular boundary homomorphism, singular homology groups, reduced homology, induced map of singular homology, exact sequence, relative chain groups, relative homology, naturality, Betti number, degree of map, local degree, cellular homology
Main Idea: Homology is kernel of a map mod the image of the preceding map; it is cycles mod boundaries. A simplex is a triangle. Simplicial homology uses simplicies, singular homology uses maps, cellular homology uses CW-complexes. Make sure you know the long exact sequence of the pair, of the triple, excision,

and anything else mentioned. This is a big, meaty section, with a lot of theorems and proofs, but the takeaway for a qualifying exam is knowing how to compute homology groups. Get some practice doing that!

Definition 2.8.1. The standard n**-simplex** is

$$\Delta^n = \left\{ (x_0, ..., x_n) \in \mathbf{R}^{n+1} \mid x_i \geq 0 \text{ and } \sum_{i=0}^{n} x_i = 1 \right\}.$$

For example, Δ^1 is a line, Δ^2 is a triangle, and Δ^3 is a tetrahedron (D4). A good basic intuition to have is that an n-simplex is a generalization of a triangle.

We remark for later that we will want to orient simplices. If a simplex is composed of vertices $[v_0, ..., v_n]$, then we orient the edges from lower-numbered vertex to higher, and the faces and up so that the orientation agrees with the edges.

Example 2.8.2. Suppose we have a 2-simplex as pictured. The vertices have been given a numbering, which forces an orientation on the edges, which forces the orientation of the 2-simplex- it is oriented so that it agrees with two of the three edges.

$$\begin{tikzcd}
 & v_2 & \\
 & \circlearrowleft & \\
v_0 \arrow[uur] \arrow[rr] & & v_1 \arrow[uul]
\end{tikzcd}$$

Definition 2.8.3. We define a **Δ-complex** thusly: given

- a collection of n-simplices of various dimensions $\{{\Delta^n}_\alpha\}$, and
- a collection of sets $\mathcal{F}_i$ consisting of faces of the same dimension,

then a Δ-complex X is the quotient of $\displaystyle\coprod_\alpha {\Delta^n}_\alpha$ obtained by identifying faces in each $\mathcal{F}_i$.

We can informally say that a Δ-complex is built by gluing simplicies together along faces.

Example 2.8.4. A Δ-complex structure on $S^1 \times S^1$ is

$$\begin{tikzcd}
v \arrow[r, "a"] & v \\
v \arrow[u, "b"] \arrow[r, "a"'] \arrow[ur, "c"] \arrow[u, phantom, shift left=6ex, "U"] & v \arrow[u, "b"'] \arrow[u, phantom, shift right=6ex, "L"]
\end{tikzcd}$$

We have two 2-simplexes, ${\Delta^2}_U$ and ${\Delta^2}_L$. Then the faces are $\mathcal{F}_a = \{a_U, a_L\}$, $\mathcal{F}_b = \{b_U, b_L\}$, and $\mathcal{F}_c = \{c_U, c_L\}$.

Definition 2.8.5. If X is a Δ-complex, then the **chain group** $\Delta_n(X)$ is the free abelian group on the set of n-simplices of x, $\{{\Delta^n}_\alpha\}$.

The elements in this group are finite formal sums:

$$\sum_\alpha n_\alpha {\Delta^n}_\alpha \in \Delta_n(X),$$

where $n_\alpha \in \mathbf{Z}$.[14]

Example 2.8.6. Let's see the chain groups of $S^1 \times S^1$ with the same Δ-complex structure as **Example 2.8.4**. There is one 0-simplex v, three 1-simplices a, b, c, and two 2-simplices U, L. We see that

$$\begin{aligned}
&\Delta_0(S^1 \times S^1) = \{n_1 v\} \cong \mathbf{Z}, \\
&\Delta_1(S^1 \times S^1) = \{n_1 a + n_2 b + n_3 c\} \cong \mathbf{Z}^3, \text{ and} \\
&\Delta_2(S^1 \times S^1) = \{n_1 U + n_2 L\} \cong \mathbf{Z}^2.
\end{aligned}$$

Definition 2.8.7. We define a **boundary homomorphism between chain groups** to be $\partial_n : \Delta_n(X) \to \Delta_{n-1}(X)$, where

$$\partial_n([v_0, v_1, ..., v_n]) = \sum_{i=0}^{n} (-1)^i [v_0, ..., \hat{v_i}, ..., v_n],$$

where $\hat{v_i}$ means to omit the ith vertex in the alternating sum.

Note that this is precisely the reason we needed to orient simplices in the definition.

Example 2.8.8. The most basic example is $\partial_1([v_0, v_1]) = [v_1] - [v_0]$. The picture is $v_0 \to v_1$.

Example 2.8.9. $\partial_2([v_0, v_1, v_2]) = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$. The picture is as in **Example 2.8.2**.

Example 2.8.10. $\partial_3([v_0, v_1, v_2, v_3]) = [v_1, v_2, v_3] - [v_0, v_2, v_3] + [v_0, v_1, v_3] - [v_0, v_1, v_2]$.

We remark the orientation of a 3-simplex does agree with the right hand rule.

Lemma 2.8.11. *The composition*

$$\Delta_n(X) \xrightarrow{\partial_n} \Delta_{n-1}(X) \xrightarrow{\partial_{n-1}} \Delta_{n-2}(X)$$

is zero; i.e., $\mathrm{Im}(\partial_n) \subseteq \ker(\partial_{n-1})$.

[14] We'll later consider arbitrary groups, not just $\mathbf{Z}$.

Physically, you can think of this as e.g. the boundary of a triangle is a circle, which has no boundary, or the boundary of a tetraderon is a sphere, which has no boundary, which is helpful for the intuition of cycles and boundaries, but often it is nice just to think of these as algebraic objects.

Proof. This is just a computation grind:

$$\begin{aligned}
\partial_{n-1}\partial_n([v_0, ..., v_n]) &= \partial_{n-1}\left(\sum_{i=0}^{n}(-1)^i[v_0, ..., \hat{v_i}, ..., v_n]\right) \\
&= \sum_{i=0}^{n}\left(\sum_{j=0}^{i-1}(-1)^i(-1)^j[v_0, ..., \hat{v_j}, ..., \hat{v_i}, ..., v_n]\right. \\
&\quad \left. + \sum_{j=i+1}^{n}(-1)^i(-1)^{j-1}[v_0, ..., \hat{v_i}, ..., \hat{v_j}, ..., v_n]\right) \\
&= \sum_{0\le j\le i\le n}(-1)^i(-1)^j[v_0, ..., \hat{v_j}, ..., \hat{v_i}, ..., v_n] \\
&\quad + \sum_{0\le i\le j\le n}(-1)^i(-1)^{j-1}[v_0, ..., \hat{v_i}, ..., \hat{v_j}, ..., v_n].
\end{aligned}$$

These sums cancel term by term, so they sum to zero. □

Definition 2.8.12. A **chain complex** is a collection $C_\bullet$ consisting of C_n, free abelian groups, and maps ∂_n such that

$$\cdots \longrightarrow C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \longrightarrow 0$$

and $\partial_n : C_n \to C_{n-1}$ is a homomorphism with $\partial_n\partial_{n-1} = 0$.

Example 2.8.13. Chain groups with boundary homomorphisms are a chain complex, by **Lemma 2.8.11**.

Definition 2.8.14. We define **homology groups** to be

$$H_n(C_\bullet) = {}^{\ker \partial_n}\!/_{\mathrm{Im}\,\partial_{n+1}},$$

where we call $\ker \partial_n$ n-cycles, and $\mathrm{Im}\,\partial_{n+1}$ boundaries. Thus we can say that homology is cycles mod boundaries. By **Lemma 2.8.11**, this is well-defined.

In the specific case when $C_n = \Delta_n(X)$, we write $H_n^\Delta(X) = H_n(C_\bullet)$, and we call this simplicial homology.[15]

[15] We'll show further into this paper that pretty much all the homologies we'll deal with in the course agree, so I may become sloppy with denoting the "Δ." I immediately become lazy about referring to the type of homology; while I may or may not write the Δ, I immediately start saying "the homology" instead of "the simplicial homology."

Example 2.8.15. Let's compute the homology groups of the wedge of m circles, $X = \bigvee_{i=1}^{m} S^1$.

Give X the Δ-complex with one vertex v and m edges $e_1, ..., e_m$. Then $\partial_1(e_i) = v - v = 0$ for all i. We also see that since $\{\Delta^0\} = \{v\}$ and $\{\Delta^1\} = \{e_1, ..., e_m\}$, we have $\Delta_0(X) \cong \mathbf{Z}$ and $\Delta_1(X) \cong \mathbf{Z}^m$. Then the chain complex is

$$\cdots 0 \xrightarrow{0} \mathbf{Z}^m \xrightarrow{0} \mathbf{Z} \xrightarrow{0} 0.$$

All maps are the zero map. Thus $\ker 0$ is the whole domain and $\operatorname{Im} 0 = 0$, so we have

$$\begin{aligned} H_0^{\Delta}(X) &= {\ker \partial_0}/{\operatorname{Im} \partial_1} = {\mathbf{Z}}/{0} = \mathbf{Z}, \\ H_1^{\Delta}(X) &= {\ker \partial_1}/{\operatorname{Im} \partial_2} = {\mathbf{Z}^m}/{0} = \mathbf{Z}^m, \text{ and} \\ H_n^{\Delta}(X) &= {\ker \partial_n}/{\operatorname{Im} \partial_{n+1}} = {0}/{0} = 0 \end{aligned}$$

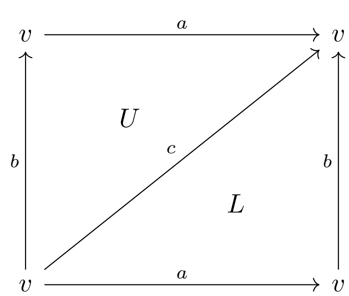
for all $n \geq 2$.

We usually like to succinctly sum up our answer at the end:

$$H_n^{\Delta}\left(\bigvee_{i=1}^{m} S^1\right) = \begin{cases} \mathbf{Z} & \text{if } n = 0, \\ \mathbf{Z}^m & \text{if } n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

We remark that nonzero homology is stating that there are cycles that are not boundaries.

Example 2.8.16. Let $X = S^1 \times S^1$ and give X the Δ-complex structure as in **Example 2.8.4**. We reproduce it below:



Again, $\partial_1 = 0$, and since there is a single 0-simplex, $H_0^{\Delta}(X) = {\ker \partial_0}/{\operatorname{Im} \partial_1} = {\mathbf{Z}}/{0} = \mathbf{Z}$.

To see ∂_2, we compute it on each 2-simplex:

$$\begin{aligned} \partial_2 U &= a + b - c, \text{ and} \\ \partial_2 L &= a + b - c. \end{aligned}$$

Then $\operatorname{Im} \partial_2 = \langle a + b - c \rangle \subseteq \Delta_1(X) \cong \mathbf{Z}^3$, and $\ker \partial_2 = \langle U - L \rangle \subseteq \Delta_2(X) \cong \mathbf{Z}^2$.

We have the chain complex:

$$\cdots \longrightarrow 0 \xrightarrow{\partial_3} \mathbf{Z}^2 \xrightarrow{\partial_2} \mathbf{Z}^3 \xrightarrow{\partial_1} \mathbf{Z} \xrightarrow{\partial_0} 0.$$

Then we have

$$\begin{aligned} H_1^{\Delta}(X) &= {}^{\ker \partial_1}\!\big/_{\operatorname{Im} \partial_2} = {}^{\mathbf{Z}^3}\!\big/_{\langle a+b-c\rangle} \\ &= {}^{\langle a, b, a+b-c\rangle}\!\big/_{\langle a+b-c\rangle} = \langle a, b\rangle = \mathbf{Z}^2, \end{aligned}$$

as $\{a, b, a+b-c\}$ is a basis for $\mathbf{Z}^3$.

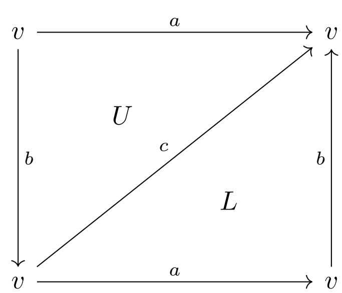
Further, we have $H_2^{\Delta}(X) = {}^{\ker \partial_2}\!\big/_{\operatorname{Im} \partial_3} = {}^{\langle U-L\rangle}\!\big/_{0} = \langle U - L\rangle = \mathbf{Z}$.

Therefore, we see that

$$H_n^{\Delta}(S^1 \times S^1) = \begin{cases} \mathbf{Z} & \text{if } n = 0, \\ \mathbf{Z}^2 & \text{if } n = 1, \\ \mathbf{Z} & \text{if } n = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.8.17. *If X is path connected, then $H_0^{\Delta}(X) = \mathbf{Z}$.*[16]

Example 2.8.18. Let's compute the homology groups of the Klein bottle. Let X be the Klein bottle, and we give a Δ-complex below.



Since $\partial_1 a = \partial_1 b = \partial_1 c = 0$ and we have one 0-simplex v, we get $H_0^{\Delta}(X) = {}^{\ker \partial_0}\!\big/_{\operatorname{Im} \partial_1} = {}^{\mathbf{Z}}\!\big/_{0} = \mathbf{Z}$.

Again we have

$$\cdots \longrightarrow 0 \xrightarrow{\partial_3} \mathbf{Z}^2 \xrightarrow{\partial_2} \mathbf{Z}^3 \xrightarrow{\partial_1} \mathbf{Z} \xrightarrow{\partial_0} 0.$$

To compute ∂_2, observe that

$$\begin{aligned} &\partial_2(U) = b + c - a, \text{ and} \\ &\partial_2(L) = a + b - c. \end{aligned}$$

These vectors are linearly independent over $\mathbf{Z}$. This implies that $\ker \partial_2 = 0$. Thus $H_2^{\Delta}(X) = {}^{\ker \partial_2}\!\big/_{\operatorname{Im} \partial_3} = {}^{0}\!\big/_{0} = 0$.

[16] We can actually upgrade this from a **Proposition**; **Theorem 2.8.29** that $H_0(X) = \mathbf{Z}$ plus the fact that the homologies agree, **Theorem 2.8.54**, gives it to us.

Finally, $\operatorname{Im}\partial_2 = \langle b+c-a, a+b-c\rangle \cong \langle b+c-a, 2b\rangle$ by just adding the two generators. Thus,

$$\begin{aligned} H_1^{\Delta}(X) &= {\langle a, bc\rangle}\big/{\langle b+c-a, 2b\rangle} \\ &= {\langle a, b, b+c-a\rangle}\big/{\langle b+c-a, 2b\rangle} \\ &= \mathbf{Z} \oplus {\mathbf{Z}}\big/{2\mathbf{Z}}. \end{aligned}$$

We conclude that

$$H_n^{\Delta}(\text{Klein bottle}) = \begin{cases} \mathbf{Z} & \text{if } n = 0, \\ \mathbf{Z} \oplus \mathbf{Z}_2 & \text{if } n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Now would be as good a time as any to make sure that the homology of a Δ-complex is actually useful. In other words, we'd like to make sure it's topologically invariant. The homology groups shouldn't depend on the Δ-complex structure we give our space. We'll describe another way to compute homology, show that *it* is topologically invariant, and then show that it is the same as our original way.

Definition 2.8.19. A **singular n-simplex** is a map $\sigma : \Delta^n \to X$.

Definition 2.8.20. A **singular n-chain** is a finite formal sum of singular n-simplices; i.e., $\displaystyle\sum_i n_i\sigma_i$.

Definition 2.8.21. We call $C_n(X)$ the **group of singular n-chains**, which is a free abelian group with basis the singular n-simplices of X. Note that this is uncountable!

Definition 2.8.22. Then the **singular boundary homomorphism** is, as before, $\partial_n : C_n(X) \to C_{n-1}(X)$, now where

$$\partial_n(\sigma) = \sum_i (-1)^i \, \sigma|_{[v_0,\ldots,\hat{v_i},\ldots,v_n]}.$$

Lemma 2.8.23. *Regarding the singular boundary homomorphism, $\partial_{n-1}\partial_n = 0$.*

Proof. Exactly as in **Lemma 2.8.11**. □

Definition 2.8.24. We define **singular homology groups** to be

$$H_n(X) = {\ker\partial_n}\big/{\operatorname{Im}\partial_{n+1}}.$$

Theorem 2.8.25. *If X and Y are homeomorphic, then $H_n(X) \cong H_n(Y)$.*

Proof. Simply compose the homeomorphism with the maps σ. □

Warning! Singular homology is highly non-computable. There's really only one space where we can compute it:

Example 2.8.26. Let's compute the singular homology of a point, $\bullet$. See that $C_n(\bullet) \cong \mathbf{Z}$ for all n, since there is a unique singular n-simplex $\sigma_n : \Delta^n \to \bullet$.

Therefore, the boundary maps are

$$\partial_n(\sigma_n) = \sum_i (-1)^i \sigma_{n-1} = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ \sigma_{n-1} & \text{if } n \text{ is even.} \end{cases}$$

Thus the chain complex is

$$\begin{tikzcd} \cdots \arrow[r, "\sim"] & \mathbf{Z} \arrow[r, "0"] \arrow[d, equal] & \mathbf{Z} \arrow[r, "\sim"] \arrow[d, equal] & \mathbf{Z} \arrow[r, "0"] \arrow[d, equal] & \mathbf{Z} \arrow[r] \arrow[d, equal] & 0 \\ \cdots \arrow[r] & C_3 \arrow[r, "\partial_3"] & C_2 \arrow[r, "\partial_2"] & C_1 \arrow[r, "\partial_1"] & C_0 \arrow[r] & 0 \end{tikzcd}$$

We have alternating isomorphisms and 0 maps.

For $n = 0$, $H_0(\bullet) = {}^{\ker \partial_0}\!/_{\operatorname{Im} \partial_1} = {}^{\mathbf{Z}}\!/_{0} = \mathbf{Z}$.

For $n \geq 1$,we see that $\ker \partial_n = \operatorname{Im} \partial_{n+1}$, so $H_n(\bullet) = {}^{\ker \partial_n}\!/_{\operatorname{Im} \partial_{n+1}} = {}^{\ker \partial_n}\!/_{\ker \partial_n} = 0$. Thus

$$H_n(\bullet) \begin{cases} \mathbf{Z} & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 2.8.27. *If X_α are the path components of X, then*

$$H_n(X) \cong \bigoplus_\alpha H_n(X_\alpha).$$

Proof. Since an n-simplex Δ^n is path connected, $\sigma : \Delta^n \to X_\alpha$ for some α. Thus, $C_n(X) \cong \bigoplus_\alpha C_n(X_\alpha)$, and ∂_n respects this splitting. $\square$

Corollary 2.8.28. *The singular homology for a discrete set $\{\bullet_\alpha\}_\alpha$ is*

$$H_n(\{\bullet_\alpha\}) \begin{cases} \bigoplus_\alpha \boldsymbol{Z} & \textit{if } n = 0, \\ 0 & \textit{otherwise.} \end{cases}$$

Theorem 2.8.29. *If $X \neq \emptyset$ and X is path connected, then $H_0(X) = \boldsymbol{Z}$.*

Proof. Note that since X is path connected $\partial_0 = 0$, so $H_0(X) = {}^{\ker \partial_0}\!/_{\operatorname{Im} \partial_1} = {}^{C_0(X)}\!/_{\operatorname{Im} \partial_1}$. Define the augmentation map $\varepsilon : C_0(X) \to \mathbf{Z}$ by $\varepsilon(\sum n_i \sigma_i) = \sum n_i$. Since $X \neq \emptyset$, ε is surjective. We will show $\ker \varepsilon = \operatorname{Im} \partial_1$.

To see that $\operatorname{Im} \partial_1 \subseteq \ker \varepsilon$, see that if $\sigma : \Delta^1 \to X$ where $\Delta^1 = [v_0, v_1]$, then $\varepsilon \partial_1(\sigma) = \varepsilon \left(\sigma|_{[v_1]} - \sigma|_{[v_0]} \right) = 1 - 1 = 0$.

To see that $\ker \varepsilon \subseteq \operatorname{Im} \partial_1$, for $\sigma_i : \Delta^0 \to X$ where $\Delta^0 = [v_0]$, suppose that $\varepsilon \left(\sum n_i \sigma_i \right) = 0$. Then we can choose paths $\tau_i : I \to X$ from some base point $x_0 \in X$ to $\sigma_i(v_0)$, as X is path connected. Then we can view $\tau_i : \Delta^1 \to X$, and so $\partial \tau_i = \sigma_i - \sigma_0$. Then $\partial_1 \left(\sum n_i \tau_i \right) = \sum n_i \sigma_i - \sum n_i \sigma_0 = \sum n_i \sigma_i$. Thus $\ker \varepsilon = \operatorname{Im} \partial_1$, as desired. $\square$

Why was this enough to show the claim? It is because of the following:

Definition 2.8.30. Consider the augmented chain complex:

$$\cdots \xrightarrow{\partial_3} C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\varepsilon} \mathbf{Z} \longrightarrow 0$$
$$\sum n_i \sigma_i \longmapsto \sum n_i$$

The homology of this chain complex is called the **reduced homology** and is written $\widetilde{H_n}(X)$. Note that the only difference between reduced and ordinary homology is in dimension 0:

$$H_0(X) = \widetilde{H_0}(X) \oplus \mathbf{Z}.$$

Corollary 2.8.31. $\widetilde{H_0}(\bullet) = 0$.

Definition 2.8.32. Like the fundamental group, if $f : X \to Y$ is a map, then the **induced map of singular homology** is the homomorphism $f_\# : C_n(X) \to C_n(Y)$ where $f_\#(\sigma) = f\sigma$ for all n.

Lemma 2.8.33. *$f_\#\partial^X = \partial^Y f_\#$; i.e., $f_\#$ is a chain map.*

Proof. We again just compute:

$$\begin{aligned} f_\#\partial\sigma &= f_\#\left(\sum_i (-1)^i \, \sigma|_{[v_0,\dots,\hat{v_i},\dots,v_n]}\right) \\ &= \sum_i \left((-1)^i \, (f\sigma)|_{[v_0,\dots,\hat{v_i},\dots,v_n]}\right) \\ &= \partial f_\#\sigma. \end{aligned}$$

□

Equivalently to **Lemma 2.8.33**, we can say that the following diagram commutes:

$$\begin{tikzcd} \cdots \arrow[r] & C_{n+1}(X) \arrow[r, "\partial_{n+1}"] \arrow[d, "f_\#"] & C_n(X) \arrow[r, "\partial_n"] \arrow[d, "f_\#"] & C_{n-1}(X) \arrow[r, "\partial_{n-1}"] \arrow[d, "f_\#"] & \cdots \\ \cdots \arrow[r] & C_{n+1}(Y) \arrow[r, "\partial_{n+1}"] & C_n(Y) \arrow[r, "\partial_n"] & C_{n-1}(Y) \arrow[r, "\partial_{n-1}"] & \cdots \end{tikzcd}$$

Theorem 2.8.34. *$f_\#$ induces a homomorphism $f_* : H_n(X) \to H_n(Y)$.*

Proof. If $\alpha \in C_n(X)$ is a cycle, then $\partial\alpha = 0$. Thus $\partial f_\#(\alpha) = f_\#\partial\alpha = 0$, so $f_\#$ maps cycles to cycles.

Further, if $\alpha \in C_n(X)$ is a boundary, then $\alpha = \partial\beta$, so $f_\#(\alpha) = f_\#\partial\beta = \partial f_\#(\beta)$, so $f_\#$ maps boundaries to boundaries. □

Again like the fundamental group, we get the following functorial properties of induced maps:

1. $(fg)_* = f_* g_*$ whenever $X \xrightarrow{g} Y \xrightarrow{f} Z$.
2. $(id_X)_* = id_{H_n(X)}$.

In category theory terms, this shows that H_n is a covariant functor from the category of topological spaces to the category of graded abelian groups.

Theorem 2.8.35. *If $f \simeq g$, then $f_* = g_*$.*

Proof. The idea is to construct a homomorphism $P : C_n(X) \to C_{n+1}(Y)$ for all n such that for all $\alpha \in C_n(X)$,

$$g_\#(\alpha) - f_\#(\alpha) = \partial P(\alpha) + P\partial(\alpha).$$

The map here is called the prism operator.

Let f be homotopic to g. Then let $F : X \times I \to Y$ be the homotopy between f and g. We define P in the following way:

For $n = 0$, take a singular 0-simplex $\sigma : \Delta^0 \to X$. Then $\Delta^0 \times I = \Delta^1$, and we map $\Delta^0 \times I \xrightarrow{\sigma \times id} X \times I \xrightarrow{F} Y$. Thus $P(\sigma) = F(\sigma \times id)$ and we have

$$\begin{aligned} \partial P(\sigma) + P\partial(\sigma) &= \partial P(\sigma) + 0 \\ &= \partial P(\sigma) \\ &= -f_\#(\sigma) + g_\#(\sigma), \end{aligned}$$

as desired.

For $n = 1$, take a singular 1-simplex $\sigma : \Delta^1 \to X$. Then $h : \Delta^1 \times I \xrightarrow{\sigma \times id} X \times I \xrightarrow{F} Y$. Note that if $\sigma = [v_0, v_1] = \sigma \times \{0\}$, then $[w_0, w_1] = \sigma \times \{1\}$, and $h([v_0, v_1]) = f_\#(\sigma)$ and $h([w_0, w_1]) = g_\#(\sigma)$.

Note also that $\Delta^1 \times I$ is not a singular n-simplex, so we decompose the square $\Delta^1 \times I$ as $[v_0, w_0, w_1]$ and $[v_0, v_1, w_1]$.

Then we have

$$\begin{aligned} P(\sigma) &= F(\sigma \times id)|_{[v_0,w_0,w_1]} - F(\sigma \times id)|_{[v_0,v_1,w_1]} \\ &= h|_{[v_0,w_0,w_1]} - h|_{[v_0,v_1,w_1]} \end{aligned}$$

and

$$\begin{aligned} \partial P(\sigma) &= \partial \left(h|_{[v_0,w_0,w_1]} - h|_{[v_0,v_1,w_1]} \right) \\ &= h|_{[w_0,w_1]} - h|_{[v_0,w_1]} + h|_{[v_0,w_0]} - h|_{[v_1,w_1]} + h|_{[v_0,w_1]} - h|_{[v_0,v_1]} \\ &= g_\#(\sigma) - f_\#(\sigma) + h|_{[v_0,w_0]} - h|_{[v_1,w_1]} \\ &= g_\#(\sigma) - f_\#(\sigma) - h|_{\text{vertical sides of } \Delta^1 \times I} \\ &= g_\#(\sigma) - f_\#(\sigma) - P\partial(\sigma). \end{aligned}$$

For the general case, let $\sigma : \Delta^n \to X$ be in $C_n(X)$. Then

$$P(\sigma) = \sum (-1)^i \; F(\sigma \times id)|_{[v_0,\ldots,v_i,w_i,\ldots,w_n]} \, .$$

For example, $\Delta^2 \times I$ decomposes into three tetrahedra.

The key idea is that applying P to a face of σ gives the corresponding side of $\partial P(\sigma)$ above the face. Thus ∂P is the top, $g_\#$, minus the bottom, $f_\#$, minus the sides, $P\partial$, of $\partial\Delta^n \times I$. While this can be formally shown, we will take it on faith that we have constructed such a map P.

P is called a chain homotopy. We claim that chain homotopic maps induces the same map on homology, which will complete the proof. To see this, let $\alpha \in C_n(X)$ be a cycle. Then $g_\#(\alpha) - f_\#(\alpha) = \partial P(\alpha) + P\partial(\alpha) = \partial P(\alpha)$. Thus, $g_\#(\alpha) - f_\#(\alpha)$ is a boundary. We conclude that $[g_\#(\alpha)] = g_*([\alpha])$ and $[f_\#(\alpha)] = f_*([\alpha])$ are equal in $H_n(Y)$, as desired. $\square$

Corollary 2.8.36. *If $f : X \to Y$ is a homotopy equivalence, then $f_* : H_n(X) \to H_n(Y)$ is an isomorphism for all n.*

Proof. The proof follows immediately from the functorial properties of the induced map and from **Theorem 2.8.35**. $\square$

Example 2.8.37. If $X \simeq \bullet$, i.e., contractible, then $\widetilde{H_n}(X) = 0$ for all n.

Definition 2.8.38. An **exact sequence** is a sequence

$$G_n \xrightarrow{f_n} G_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} G_0$$

such that $\operatorname{Im} f_{n+1} = \ker f_n$.

If we have $0 \to A \to B \to C \to 0$ exact, we call it a **short exact sequence**.

A **long exact sequence** is an exact sequence of infinite length.

Proposition 2.8.39. *If X is a space and $A \subseteq X$ is a subspace such that there exists $U \subseteq X$ open that deformation retracts onto A, then there exists a long exact sequence*

$$\begin{tikzcd}
\cdots \arrow[r] & \widetilde{H_n}(A) \arrow[r, "i_*"] & \widetilde{H_n}(X) \arrow[r, "j_*"] & \widetilde{H_n}\left(X/A\right) \arrow[dll, "\partial"] \\
& \widetilde{H_{n-1}}(A) \arrow[r, "i_*"] & \widetilde{H_{n-1}}(X) \arrow[r] & \cdots
\end{tikzcd}$$

i.e., $\operatorname{Im} i_* = \ker j_*$, $\operatorname{Im} j_* = \ker \partial$, *and* $\operatorname{Im} \partial = \ker i_*$.

We remark that though we don't prove **Proposition 2.8.39** here, the tools built up with relative homology do the trick. See **Theorem 2.8.50**.

Corollary 2.8.40.

$$\widetilde{H_k}(S^n) \begin{cases} \boldsymbol{Z} & \textit{if } k = n, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. We use **Proposition 2.8.39** with $(X, A) = (D^n, S^{n-1})$. Then ${}^{X}\!/\!_{A} = {}^{D^n}\!/\!_{S^{n-1}} = S^n$.

Since $\widetilde{H_k}(X) = \widetilde{H_k}(D^n) = 0$ for all k by **Example 2.8.37**, we have the short exact sequences

$$0 \xrightarrow{j_*} \widetilde{H_k}\left({}^{X}\!/\!_{A}\right) \xrightarrow{\partial} \widetilde{H_{k-1}}(A) \xrightarrow{i_*} 0$$

We see that when we have an exact sequence of the form $0 \to A \to B \to 0$, then $A \cong B$.

> Since the sequence is exact, $\mathrm{Im}(A \to B) = \ker(B \to 0) = B$, so $A \to B$ is surjective. Furthermore $\ker(A \to B) = \mathrm{Im}(0 \to A) = 0$, so $A \to B$ is injective. Thus $A \to B$ is a bijective homomorphism, i.e., an isomorphism.

Therefore, $\widetilde{H_k}(S^n) \cong \widetilde{H_{k-1}}(S^{n-1})$. Since

$$\widetilde{H_k}(S^0) \begin{cases} \mathbf{Z} & \text{if } k = 0, \\ 0 & \text{otherwise,} \end{cases}$$

induction proves the corollary. $\square$

Now we can prove an old theorem for the more general case. We recall **Brouwer's Fixed Point Theorem 1.55.6**; we now show it holds for any $n > 2$.

Theorem 2.8.41 (**Brouwer's Fixed Point Theorem**)**.** *Any continuous map $f : D^n \to D^n$ has a fixed point.*

Proof. Suppose not. Then there exists a retraction $r : D^n \to S^{n-1}$. This gives

$$\begin{tikzcd} \mathbf{Z} = H_{n-1}(S^{n-1}) \arrow[rr, "id"] \arrow[dr, bend right, "i_*"'] & & H_{n-1}(S^{n-1}) = \mathbf{Z} \\ & H_{n-1}(D^n) = 0 \arrow[ur, bend right, "r_*"'] & \end{tikzcd}$$

This is a contradiction. $\square$

Definition 2.8.42. If $A \subseteq X$, define the **relative chain groups** to be

$$C_n(X, A) = {}^{C_n(X)}\!/\!_{C_n(A)}.$$

We won't use this for a while, but if X is a Δ-complex and A is a subcomplex, we can also define relative chain groups here:

$$\Delta_n(X, A) = {}^{\Delta_n(X)}\!/\!_{\Delta_n(A)}$$

Definition 2.8.43. Since $\partial(C_n(A)) \subseteq C_{n-1}(A)$, we get an induced chain complex

$$\cdots \xrightarrow{\partial} C_{n+1}(X,A) \xrightarrow{\partial} C_n(X,A) \xrightarrow{\partial} C_{n-1}(X,A) \xrightarrow{\partial} \cdots$$

Thus we can define **relative homology** to be

$$H_n(X,A) = {}^{\ker \partial_n}\!\big/_{\operatorname{Im} \partial_{n+1}}.$$

And we can do the same for Δ-complexes:

$$H_n^{\Delta}(X,A) = {}^{\ker \partial_n}\!\big/_{\operatorname{Im} \partial_{n+1}}.$$

We remark that in relative homology, $\ker \partial$ are the relative cycles; i.e., the $\alpha \in C_n(X)$ such that $\partial\alpha \in C_{n-1}(A)$. Furthermore, $\operatorname{Im} \partial$ are the relative boundaries, which can be thought of as $\alpha \in C_n(X)$ such that $\alpha = \partial\beta + \gamma$ where $\beta \in C_{n+1}(X)$ and $\gamma \in C_n(A)$.

We return to **Proposition 2.8.39**, suitably restated in the language of relative homology:

Theorem 2.8.44. *If (X,A) is a pair with $A \subseteq X$, then there exists a long exact sequence*

$$\begin{tikzcd}
\cdots \arrow[r] & \widetilde{H}_n(A) \arrow[r, "i_*"] & \widetilde{H}_n(X) \arrow[r, "j_*"] & \widetilde{H}_n(X,A) \arrow[dll, "\partial"] \\
& \widetilde{H_{n-1}}(A) \arrow[r, "i_*"] & \widetilde{H_{n-1}}(X) \arrow[r] & \cdots
\end{tikzcd}$$

Proof. This follows from the short exact sequence of chain complexes:

$$\begin{tikzcd}
0 \arrow[r] & C_\bullet(A) \arrow[r, "i"] \arrow[d, equal] & C_\bullet(X) \arrow[r, "j"] \arrow[d, equal] & C_\bullet(X,A) \arrow[r] \arrow[d, equal] & 0 \\
& \bigoplus_n C_n(A) & \bigoplus_n C_n(X) & \bigoplus_n C_n(X,A) &
\end{tikzcd}$$

Call this exact sequence $0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$. Then the following diagram commutes:

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & A_{n+1} \arrow[r] \arrow[d, "i"] & A_n \arrow[r] \arrow[d, "i"] & A_{n-1} \arrow[r] \arrow[d, "i"] & \cdots \\
\cdots \arrow[r] & B_{n+1} \arrow[r] \arrow[d, "j"] & B_n \arrow[r] \arrow[d, "j"] & B_{n-1} \arrow[r] \arrow[d, "j"] & \cdots \\
\cdots \arrow[r] & C_{n+1} \arrow[r] \arrow[d] & C_n \arrow[r] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & \cdots \\
& 0 & 0 & 0 &
\end{tikzcd}$$

We get $i_* : H_n(A_\bullet) \to H_n(B_\bullet)$ induced from the chain map i and $j_* : H_n(B_\bullet) \to H_n(C_\bullet)$ induced from the chain map j. We just need ∂ now. A nice remark for algebraists is that what follows is just the **Snake Lemma**.

Let's define $\partial : H_n(C_\bullet) \to H_{n-1}(A_\bullet)$. Let $c \in C_n$ be a cycle; i.e., $\partial c = 0$.

$$\begin{tikzcd}
 & & a \arrow[d, mapsto] & A_{n-1} \arrow[dd, "i"] \\
b \arrow[rr, mapsto] \arrow[dd, mapsto] & & \partial b & \\
 & B_n \arrow[rr, "\partial"] \arrow[d, "j"] & & B_{n-1} \\
c & C_n & &
\end{tikzcd}$$

As j is surjective, there exists $b \in B_n$ such that $j(b) = c$. So $\partial b \in B_{n-1}$ and $j(\partial b) = \partial j(b) = \partial(c) = 0$. So $\partial b \in \ker j = \operatorname{Im} i$, so $\partial b = i(a)$ for some $a \in A_{n-1}$. Now, $\partial a = 0$ if and only if $i(\partial a) = 0$, as i is injective, since $\ker i = 0$. But $i\partial(a) = \partial i(a) = \partial^2(b) = 0$, as this is a chain complex. So a is in fact a cycle, and thus we define $\partial[c] = [a]$.

We now must ask: is this well-defined? We had to choose c, b, and a, so we hope that our definition doesn't depend on those choices. We answer yes. To see this,

- a is uniquely determined given ∂b, since i is injective.
- If $j(b) = j(b')$, then $j(b - b') = 0$, since j is a homomorphism. Thus $b - b' \in \ker j = \operatorname{Im} i$. Therefore $b' = b + i(a')$ for some $a' \in A_n$. Then $i(a + \partial a') = i(a) + \partial i(a') = \partial b + \partial(b' - b) = \partial b'$. As $[a] = [a + \partial a']$, changing b doesn't affect ∂.
- If we replace c by $c + \partial c'$, then we have $c + \partial c' = c + \partial j(b')$ for some $b' \in B_{n+1}$. So $c + \partial j(b') = c + j(\partial b') = j(b) + j(\partial b') = j(b + \partial b')$, and as $\partial(b + \partial b') = \partial b$, changing c in $[c]$ doesn't affect ∂.

To confirm that the sequence is exact is just diagram chasing, so we defer it. $\square$

We describe the geometric sense of $\partial : H_n(X, A) \to H_{n-1}(A)$, as we did for relative cycles and relative boundaries. Let $[\alpha] \in H_n(X, A)$ be represented by a relative cycle $\alpha \in C_n(X)$. Then $\partial[\alpha]$ is the class of $\partial\alpha \in C_{n-1}(A)$.

Proposition 2.8.45. *The relative reduced homology is isomorphic to the relative homology; i.e.,* $\widetilde{H_n}(X, A) \cong H_n(X, A)$ *for all* n.

Lemma 2.8.46. *Let* X *be a space and* $x_0 \in X$ *be a point. Then* $\widetilde{H_n}(X) \cong H_n(X, x_0)$.

Proof. Using the long exact sequence of **Theorem 2.8.44**, since $\widetilde{H_n}(x_0) = 0$ for all n, we have $0 \to \widetilde{H_n}(X) \to \widetilde{H_n}(X, x_0) \to 0$ for all n. Then $\widetilde{H_n}(X) \cong \widetilde{H_n}(X, x_0)$. By **Proposition 2.8.45**, $\widetilde{H_n}(X) \cong H_n(X, x_0)$, as desired. $\square$

Proposition 2.8.47. *If* $f, g : (X, A) \to (Y, B)$ *are homotopic through maps of pairs, then* $f_* = g_* : H_n(X, A) \to H_n(Y, B)$.

Proposition 2.8.48. *Analogous to a pair (X, A), if given a triple $B \subseteq A \subseteq X$, then the short exact sequence*

$$0 \longrightarrow C_\bullet(A, B) \longrightarrow C_\bullet(X, B) \longrightarrow C_\bullet(X, A) \longrightarrow 0$$

gives rise to a long exact sequence

$$\begin{tikzcd} \cdots \arrow[r] & H_n(A,B) \arrow[r] & H_n(X,B) \arrow[r] & H_n(X,A) \arrow[dll, "\partial"] \\ & H_{n-1}(A,B) \arrow[r] & H_{n-1}(X,B) \arrow[r] & \cdots \end{tikzcd}$$

Theorem 2.8.49 (Excision). *Given $Z \subseteq A \subseteq X$ such that $\overline{Z} \subseteq \mathrm{Int}(A)$, then the inclusion $(X \setminus Z, A \setminus Z) \hookrightarrow (X, A)$ induces isomorphisms $H_n(X \setminus Z, A \setminus Z) \to H_n(X, A)$ for all n.*

Equivalently, one may say that for $A, B \subseteq X$ where $X = \mathrm{Int}(A) \cup \mathrm{Int}(B)$, $(B, A \cap B) \hookrightarrow (X, A)$ induces isomorphisms $H_n(B, A \cap B) \to H_n(X, A)$ for all n. (To see the equivalence, let $B = X \setminus Z$.)

Proof. Let $U = \{U_i\}$ be the collection of subsets of X such that $X = \bigcup \mathrm{Int}(U_i)$. Then ${C_n}^U \subseteq C_n(X)$ is the subgroup of n-chains $\sum n_i \sigma_i$ where, for all i, $\sigma_i(\Delta^n) \subseteq U_j$ for some j. ${C_n}^U$ are the n-chains relative to the cover U.

See that with ∂, we have a chain complex:

$$\cdots \longrightarrow {C_{n+1}}^U(X) \xrightarrow{\partial} {C_n}^U(X) \xrightarrow{\partial} {C_{n-1}}^U(X) \longrightarrow \cdots$$

Then we can define the homology relative to the cover,

$${H_n}^U(X) = {}^{\ker \partial_n}\!/\!_{\mathrm{Im}\, \partial_{n+1}}.$$

We remark now that the inclusion map $\iota : {C_n}^U(X) \to C_n(X)$ induces an isomorphism $\iota_* : {H_n}^U(X) \to H_n(X)$. Specifically, there exists a map $\rho : C_n(X) \to {C_n}^U(X)$ such that $\rho\iota = id$ and $\iota\rho$ is chain homotopic to id. The proof is technical, requiring barycentric subdivision. The general idea is that, given e.g. a singular 2-chain in $A \cup B$, we can subdivide the triangle so that it is a union of smaller triangles such that every triangle is entirely in A or B. We take this claim on faith, and continue the proof of excision.

Suppose $U = \{A, B\}$; i.e., we prove the equivalent version stated in the theorem. Then the maps ι, ρ, and the chain homotopy all send chains in A to chains in A, so we get induced maps on the quotients:

i.e., induced from ι, we have ${}^{{C_n}^U(X)}\!/\!_{C_n(A)} \to {}^{C_n(X)}\!/\!_{C_n(A)}$.

The induced maps give an isomorphism on the level of homology; in other words, ${H_n}^U(X, A) \xrightarrow{\sim} H_n(X, A)$. We remark that ${}^{C_n(B)}\!/\!_{C_n(A \cap B)}$ describes chains in B whose image is not in A, so since ${C_n(B)}/{C_n(A \cap B)} \to {C_n{}^U(X)}/{C_n(A)}$ is an isomorphism, as both are free, with basis singular n-simplices in B with image not in A, we get that $H_n(B, A \cap B) \cong H_n{}^U(X, A)$, as desired. $\square$

Theorem 2.8.50. *Suppose $A \subseteq X$ is closed and there exists $U \subseteq X$ open that deformation retracts onto A. Then the quotient map $q : (X, A) \to \left({X}/{A}, {A}/{A}\right)$ induces isomorphisms $H_n(X, A) \xrightarrow{\sim} H_n\left({X}/{A}, {A}/{A}\right) \cong \widetilde{H}_n\left({X}/{A}\right)$.*

We remark that this is enough to prove **Proposition 2.8.39**.

Proof. We use excision. Consider the diagram; we claim it commutes:

$$\begin{tikzcd}
H_n(X,A) \arrow[r] \arrow[d, "q_*", "\wr"'] & H_n(X,U) \arrow[d, "q_*", "\wr"'] & H_n(X \setminus A, U \setminus A) \arrow[l, "\text{excision}", "\sim"'] \arrow[d, "q_*"] \\
H_n\left({X}/{A}, {A}/{A}\right) \arrow[r] & H_n\left({X}/{A}, {U}/{A}\right) & H_n\left({X}/{A} \setminus {A}/{A}, {U}/{A} \setminus {A}/{A}\right) \arrow[l, "\text{excision}", "\sim"']
\end{tikzcd}$$

Consider the triple $A \subseteq U \subseteq X$. Then we have

$$H_n(U,A) \longrightarrow H_n(X,A) \longrightarrow H_n(X,U) \longrightarrow H_{n-1}(U,A).$$

As (U, A) has the same homotopy type as (A, A); i.e., $(U, A) \simeq (A, A)$ since U deformation retracts onto A, we have $H_n(U, A) \cong H_n(A, A) = 0$.

Thus what remains is

$$0 \longrightarrow H_n(X,A) \longrightarrow H_n(X,U) \longrightarrow 0.$$

Therefore $H_n(X, A) \cong H_n(X, U)$.

Thus our diagram is now

$$\begin{tikzcd}
H_n(X,A) \arrow[r, "\sim", "\text{long exact}"'] \arrow[d, "q_*", "\wr"'] & H_n(X,U) \arrow[d, "q_*", "\wr"'] & H_n(X \setminus A, U \setminus A) \arrow[l, "\text{excision}", "\sim"'] \arrow[d, "q_*"] \\
H_n\left({X}/{A}, {A}/{A}\right) \arrow[r, "\sim", "\text{long exact}"'] & H_n\left({X}/{A}, {U}/{A}\right) & H_n\left({X}/{A} \setminus {A}/{A}, {U}/{A} \setminus {A}/{A}\right) \arrow[l, "\text{excision}", "\sim"']
\end{tikzcd}$$

All that remains to be seen is that the final q_* is an isomorphism. To see this, since $q : (X \setminus A, U \setminus A) \to \left({X}/{A} \setminus {A}/{A}, {U}/{A} \setminus {A}/{A}\right)$ is a homeomorphism, therefore the map $q_* : H_n(X \setminus A, U \setminus A) \to H_n\left({X}/{A} \setminus {A}/{A}, {U}/{A} \setminus {A}/{A}\right)$ is an isomorphism.

Thus everything is an isomorphism, and the diagram commutes, as claimed. $\square$

Corollary 2.8.51. *Let $(D^n, \partial D^n)$ be the pair of an n-disk and its boundary (i.e., S^{n-1}). Consider $(D^n, \partial D^n)$ as $(\Delta^n, \partial\Delta^n)$. Recall from **Corollary 2.8.40** that $H_n(\Delta^n, \partial\Delta^n) = \boldsymbol{Z}$. We claim that the identity map $i_n : \Delta^n \to \Delta^n \in C_n(\Delta^n)$ is a cycle that generates $H_n(\Delta^n, \partial\Delta^n)$.*

Proof. Since $\partial i_n(\Delta^n) \subseteq \partial\Delta^n$ and we are using relative homology, we have i_n is a relative cycle, as claimed.

To show that i_n is a generator for all n, we proceed by induction on n.

For the base case, let $n = 0$. Then there exists a unique map $\Delta^0 \to \Delta^0$, so the claim is true.

Now, for the inductive step, assume the claim holds for $n-1$; we show it holds for n. Let $\Lambda \subseteq \partial\Delta^n$ be the union of all but one $(n-1)$-dimensional face. Now consider the triple $\Lambda \subseteq \partial\Delta^n \subseteq \Delta^n$.

We have via **Proposition 2.8.47** that $H_n(\Delta^n, \Lambda) = H_n(\Lambda, \Lambda) = 0$, as $(\Delta^n, \Lambda) \simeq (\Lambda, \Lambda)$ via deformation retraction. We also have $H_{n-1}(\Delta^n, \Lambda) = 0$ induced by including $i_{n-1} : \Delta^{n-1} \to \partial\Delta^n$ in the face not in Λ. Thus we have

$$0 = H_n(\Delta^n, \Lambda) \longrightarrow H_n(\Delta^n, \partial\Delta^n) \longrightarrow H_{n-1}(\partial\Delta^n, \Lambda) \longrightarrow H_{n-1}(\Delta^n, \Lambda) = 0.$$

Therefore $H_n(\Delta^n, \partial\Delta^n) \cong H_{n-1}(\partial\Delta^n, \Lambda)$.

Further, we have $H_{n-1}(\partial\Delta^n, \Lambda) \cong H_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1})$, since we know that $\Delta^{n-1} / \partial\Delta^{n-1} = S^{n-1} = \partial\Delta^n / \Lambda$.

Ultimately, $H_n(\Delta^n, \partial\Delta^n) \cong H_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1})$. By inductive hypothesis, we therefore conclude that $i_n \to \partial i_n = \pm i_{n-1} \in C_{n-1}(\Delta^{n-1})$. ☐

Definition 2.8.52. Given a map of pairs $f : (X, A) \to (Y, B)$, we say that the long exact sequence is **natural**[17] if the following diagram commutes.

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(A) \arrow[r] \arrow[d, "f_*"] & H_n(X) \arrow[r] \arrow[d, "f_*"] & H_n(X, A) \arrow[r] \arrow[d, "f_*"] & H_{n-1}(A) \arrow[r] \arrow[d, "f_*"] & \cdots \\
\cdots \arrow[r] & H_n(B) \arrow[r] & H_n(Y) \arrow[r] & H_n(Y, B) \arrow[r] & H_{n-1}(B) \arrow[r] & \cdots
\end{tikzcd}$$

Proposition 2.8.53 (Five Lemma). *If we have two exact sequences; i.e., each row of the following diagram is exact:*

$$\begin{tikzcd}
A_1 \arrow[r] \arrow[d, "\alpha"] & B_1 \arrow[r] \arrow[d, "\beta"] & C_1 \arrow[r] \arrow[d, "\gamma"] & D_1 \arrow[r] \arrow[d, "\delta"] & E_1 \arrow[d, "\varepsilon"] \\
A_2 \arrow[r] & B_2 \arrow[r] & C_2 \arrow[r] & D_2 \arrow[r] & E_2
\end{tikzcd}$$

then if $\alpha, \beta, \delta, \varepsilon$ are isomorphisms, then so is γ.

Now recall that we defined relative homology of Δ-complexes back in **Definition 2.8.43**. We're finally going to use it! Our goal is to show that our homology theories are the same; i.e., that $H_\bullet(X)$ and $H_\bullet^\Delta(X)$ are isomorphic.

[17] Naturality is such a common term in mathematics that general definitions may not be referring to this specifically, but in context, naturality usually means "niceness" in terms of properties that you want, and/or that diagrams commute.

Theorem 2.8.54. *Consider the map $f : \Delta_n(X) \to C_n(X)$ defined by sending a simplex $\Delta_\alpha \subseteq X$ to its characteristic map $\sigma : \Delta^n \to X$.*

This map sends $\Delta_n(A)$ to $C_n(A)$, so it induces a chain map from relative simplicial complexes to relative singular chains $\Delta_n(X, A) \to C_n(X, A)$. Hence it induces a map on homology $f_* : H_n^\Delta(X, A) \to H_n(X, A)$.

We claim f_* is an isomorphism.

Proof. Suppose first $A = \emptyset$. Then consider the pairs (X^k, X^{k-1}). We have

$$\begin{tikzcd}
H^\Delta_{n+1}(X^k, X^{k-1}) \arrow[r] \arrow[d, "1"] & H^\Delta_n(X^{k-1}) \arrow[r] \arrow[d, "2"] & H^\Delta_n(X^k) \arrow[r] \arrow[d, "3"] & H^\Delta_n(X^k, X^{k-1}) \arrow[r] \arrow[d, "4"] & H^\Delta_{n-1}(X^{k-1}) \arrow[d, "5"] \\
H_{n+1}(X^k, X^{k-1}) \arrow[r] & H_n(X^{k-1}) \arrow[r] & H_n(X^k) \arrow[r] & H_n(X^k, X^{k-1}) \arrow[r] & H_{n-1}(X^{k-1})
\end{tikzcd}$$

We could see that maps 2 and 5 are isomorphisms by induction; the base case when $k = 1$ and hence $X = \bullet$ is easy.

Now, let's consider maps 1 and 4. Consider the relative simplicial chain groups $\Delta_n(X^k, X^{k-1})$. We see that $\Delta_n(X^k, X^{k-1}) = 0$ when $n \neq k$, and when $n = k$, it is a free abelian group with basis of k-simplices. From this we can complude that $H_n^\Delta(X^k, X^{k-1})$ has the same description; 0 when $n \neq k$ and free abelian when $n = k$.

Next,

$$\begin{aligned}
X^k \diagup X^{k-1} &= \amalg {\Delta^k}_\alpha \diagup \amalg \partial {\Delta^k}_\alpha \\
&= \bigvee_\alpha {\Delta^k}_\alpha \diagup \partial {\Delta^k}_\alpha \\
&= \bigvee_\alpha {S^k}_\alpha
\end{aligned}$$

From this we can conclude $H_n(X^k, X^{k-1}) \cong \widetilde{H_n}\left(\bigvee_\alpha {\Delta^k}_\alpha \diagup \partial {\Delta^k}_\alpha\right)$. Now we see that

$$
\begin{array}{cc}
\widetilde{H_n}\left(\coprod {}^{\partial\Delta^k{}_\alpha}\!\big/_{\partial\Delta^k{}_\alpha}\right) & = \widetilde{H_n}(\coprod \bullet) = 0 \text{ (if } n \neq 0) \\
\downarrow & \\
\widetilde{H_n}\left(\coprod {}^{\Delta^k{}_\alpha}\!\big/_{\partial\Delta^k{}_\alpha}\right) & \\
\downarrow & \\
H_n\left(\coprod {}^{\Delta^k{}_\alpha}\!\big/_{\partial\Delta^k{}_\alpha}, \coprod {}^{\partial\Delta^k{}_\alpha}\!\big/_{\partial\Delta^k{}_\alpha}\right) & \\
\downarrow & \\
\widetilde{H_{n-1}}\left(\coprod {}^{\partial\Delta^k{}_\alpha}\!\big/_{\partial\Delta^k{}_\alpha}\right) & = \widetilde{H_n}(\coprod \bullet) = 0 \text{ (if } n-1 \neq 0)
\end{array}
$$

Thus we can see that if $n \neq 0, 1$, then

$$
\widetilde{H_n}\left(\coprod {}^{\Delta^k{}_\alpha}\!\big/_{\partial\Delta^k{}_\alpha}\right) \cong H_n\left(\coprod {}^{\Delta^k{}_\alpha}\!\big/_{\partial\Delta^k{}_\alpha}, \coprod {}^{\partial\Delta^k{}_\alpha}\!\big/_{\partial\Delta^k{}_\alpha}\right).
$$

Further,

$$
\begin{aligned}
\widetilde{H_n}\left(\coprod {}^{\Delta^k{}_\alpha}\!\big/_{\partial\Delta^k{}_\alpha}\right) &= \bigoplus_\alpha \widetilde{H_n}\left({}^{\Delta^k{}_\alpha}\!\big/_{\partial\Delta^k{}_\alpha}\right) \\
&= \bigoplus_\alpha H_n(\Delta^k{}_\alpha, \partial\Delta^k{}_\alpha).
\end{aligned}
$$

So we conclude that $H_n(X^k, X^{k-1}) = \bigoplus\limits_\alpha H_n(\Delta^k{}_\alpha, \partial\Delta^k{}_\alpha)$. In other words, $H_n(X^k, X^{k-1})$ is free abelian with basis k-simplices, and each $H_n(\Delta^k{}_\alpha, \partial\Delta^k{}_\alpha)$ in the direct sum is generated by identity maps. But that was exactly the simplicial homology! Therefore, maps 1 and 4, $H_n^\Delta(X^k, X^{k-1}) \to H_n(X^k, X^{k-1})$, are isomorphisms as well.

By the **Five Lemma 2.8.53**, map 3, $H_n^\Delta(X^k) \to H_n(X^k)$, is an isomorphism. Therefore, in the case that $A = \emptyset$, we are done!

The case when $A \neq \emptyset$ is actually easy now. We see that

$$
\begin{array}{ccccccccc}
H_n^\Delta(A) & \to & H_n^\Delta(X) & \to & H_n^\Delta(X,A) & \to & H_{n-1}^\Delta(A) & \to & H_{n-1}^\Delta(X) \\
\downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle 3} & & \downarrow{\scriptstyle 4} & & \downarrow{\scriptstyle 5} \\
H_n & \longrightarrow & H_n(X) & \to & H_n(X,A) & \to & H_{n-1}(A) & \to & H_{n-1}(X)
\end{array}
$$

We have already shown in the $A = \emptyset$ case that maps 1, 2, 4, and 5 are isomorphisms, so again by the **Five Lemma 2.8.53**, 3 is. We are done! □

From **Theorem 2.8.54**, we can conclude that $H_n^\Delta(X) = H_n(X)$. Hooray!

Corollary 2.8.55. *If X is a finite Δ-complex, then $H_n(X)$ is finitely generated for all n, and $H_n(X) = 0$ for $n > \dim(X)$. Thus $H_n(X) = \mathbb{Z}^{b_n} \oplus G$, where G is a finite abelian group.*

Definition 2.8.56. The number b_n such that $H_n(X) = \mathbf{Z}^{b_n} \oplus G$ is called the **nth Betti number of X**.

Definition 2.8.57. The **degree** of a map $f : S^n \to S^n$ is the integer d such that $f_*(\alpha) = d\alpha$ for all $\alpha \in H_n(S^n) \cong \mathbf{Z}$. We write $\deg(f)$.

Lemma 2.8.58. *The degree has the following properties:*

1. $\deg(id_{S^n}) = 1$,
2. $\deg(f) = 0$ *if f is not surjective,*
3. $\deg(f) = \deg(g)$ *if and only if $f \simeq g$,*
4. $\deg(fg) = \deg(f)\deg(g)$,
5. $\deg(f) = -1$ *if f is a reflection across a subsphere S^{n-1}; e.g. fix the equator S^1 of S^2 and f interchanges the two hemispheres,*
6. *if $f : S^n \to S^n$ is the antipodal map (i.e., $f(x) = -x$), then* $\deg(f) = (-1)^{n+1}$*, and*
7. *if $f(x) \neq x$ for all $x \in S^n$, then* $\deg(f) = (-1)^{n+1}$.

Proof. Let's go in order:

1. To see $\deg(id_{S^n}) = 1$, we see that $(id_{S^n})_* = id_{H_n(S^n)}$ and $id_{H_n(S^n)}(\alpha) = \alpha = 1\alpha$, so $d = 1$.

2. To see $\deg(f) = 0$ if f is not surjective, we see that if $x_0 \notin f(S^n)$, then we can factor f through the complement:

$$f : S^n \xrightarrow{f_1} S^n \setminus \{x_0\} \overset{\iota}{\hookrightarrow} S^n, \text{ and on the level of homology,}$$
$$f_* : H_n(S^n) \xrightarrow{(f_1)_*} H_n(S^n \setminus \{x_0\}) \overset{\iota_*}{\hookrightarrow} H_n(S^n).$$

   But $H_n(S^n \setminus \{x_0\})$ is 0 because $S^n \setminus \{x_0\}$ is contractible (**Example 2.8.37**). And the zero map clearly has $\deg(0) = 0$.

3. To see $\deg(f) = \deg(g)$ if and only if $f \simeq g$, we see that in one direction $f \simeq g$ implies that $f_* = g_*$, so $\deg(f) = \deg(g)$. The other direction is highly nontrivial, but if you want to hunt down a reference, Hopf proved it in 1925.

4. To see $\deg(fg) = \deg(f)\deg(g)$, we see that $(fg)_* = f_*g_*$. Thus $\deg(fg)$ is the d such that $f_*g_*(\alpha) = f_*(d_g\alpha) = d_f d_g \alpha$; i.e., $d = d_f d_g = \deg(f)\deg(g)$.

   Also we remark now that this implies that if f is a homotopy equivalence, then $\deg(f) = \pm 1$, since $fg \simeq id$ implies $\deg(f)\deg(g) = \deg(id) = 1$.

5. To see $\deg(f) = -1$ is f is a reflection across a subsphere S^{n-1}, we see that

$$S^n = {\Delta^n}_1 \bigcup_{\partial {\Delta^n}_1 = \partial {\Delta^n}_2} {\Delta^n}_2$$

implies that $0 \neq [{\Delta^n}_1 - {\Delta^n}_2] \in H_n^{\Delta}(S^n) \cong H_n(S^n)$. Then $f_*([{\Delta^n}_1 - {\Delta^n}_2]) = [{\Delta^n}_2 - {\Delta^n}_1] = -[{\Delta^n}_1 - {\Delta^n}_2]$.

6. To see if $f : S^n \to S^n$ is the antipodal map, then $\deg(f) = (-1)^{n+1}$, we see that f is the composition of $n+1$ reflections; i.e., $f : (x_1, ..., x_{n+1}) \mapsto (-x_1, x_2, ..., x_{n+1}) \mapsto (-x_1, -x_2, ..., x_{n+1}) \mapsto ... \mapsto (-x_1, -x_2, ... - x_{n+1})$. By 4. and 5., $\deg(f) = (-1)^{n+1}$.

7. To see if $f(x) \neq x$ for all $x \in S^n$, then $\deg(f) = (-1)^{n+1}$, we see that it must be the case that f is homotopic to the antipodal map, and thus by 3. and 6., we have it. We give the explicit homotopy:

$$f_t(x) = \frac{[(1-t)f(x) - tx]}{|(1-t)f(x) - tx|}.$$

This is well defined since $(1-t)f(x) - tx \neq 0$. At time $t = 0$, we have f, and at time $t = 1$, we have the antipodal map. $\square$

Theorem 2.8.59 (**Hairy Ball Theorem**). *S^n has a continuous nonzero tangent vector field if and only if n is odd.*

Proof. If $n = 2k - 1$, then we explicitly give it: $v(x_1, x_2, ..., x_{2k-1}, x_{2k}) = (-x_2, x_1, -x_4, x_3, ..., -x_{2k}, x_{2k-1})$.

Conversely, suppose $v(x)$ is a nonzero tangent vector field on S^n. Normalize it so that $|v(x)| = 1$. Then $f_t(x) = (\cos t)x + (\sin t)v(x)$ lies in the unit circle of the plane spanned by x and $v(x)$ (well-defined since $v(x)$ is tangent). As t goes from 0 to π, we get a homotopy from the identity map to the antipodal map, which implies $(-1)^{n+1} = 1$; i.e., n is odd. $\square$

Lemma 2.8.60. *If n is even, then $\mathbf{Z}_2$ is the only group that can act freely by homeomorphisms on S^n.*

Recall **Definition 2.6.13**; an action of G on X is a homomorphism $G \to \mathrm{Aut}(X)$. The action is free if the homeomorphism corresponding to every nontrivial element of G has no fixed points.

Proof. We remark that since a homeomorphism must have degree ± 1, degree defines a homomorphism from $G \to \{-1, 1\}$. Since the group acts freely, there are no fixed points; hence $\deg(f) = (-1)^{n+1}$ by number 7. in **Lemma 2.8.58**. If n is even, $\deg(f) = -1$, and the kernel of $G \to \{-1, 1\}$ is trivial; thus $G \subseteq \mathbf{Z}_2$, so $G = \mathbf{Z}_2$ or G is trivial. $\square$

Now a question to ask is: how do we compute the degree of a map in general?

To answer this question, we fix a $y \in S^n$ such that $f^{-1}(y) = \{x_1, ..., x_m\}$. Then there is a neighborhood U_i of each x_i in the preimage of y such that $f(U_i \setminus \{x_i\}) \subseteq V \setminus \{y\}$; i.e., there is only one x_i in each U_i. We remark that $(U_i, U_i \setminus \{x_i\}) \simeq (D^n, \partial D^n)$. Then we have the following monster of a diagram:

$$\begin{tikzcd}
& \mathbf{Z} \arrow[d, equal] & \mathbf{Z} \arrow[d, equal] \\
& H_n(U_i, U_i \setminus \{x_i\}) \arrow[r, "f_*"] \arrow[dl, "\sim", "\text{excision}"'] \arrow[d, "k_i"] & H_n(V, V \setminus \{y\}) \arrow[d, "\text{excision}"', "\wr"] \\
H_n(S^n, S^n \setminus \{x_i\}) & H_n(S^n, S^n \setminus f^{-1}(y)) \arrow[l, "p_i"'] \arrow[r, "f_*"] & H_n(S^n, S^n \setminus \{y\}) \\
& H_n(S^n) \arrow[r, "f_*"] \arrow[ul, "\sim"', "\text{long exact}"] \arrow[u, "j"] & H_n(S^n) \arrow[u, "\text{long exact}", "\wr"']
\end{tikzcd}$$

The maps k_i and p_i are induced by inclusion, hence the triangles commute.

The two isomorphisms from excision can be verified using **Excision 2.8.49** where $X = S^n$, $A = S^n \setminus \{\bullet\}$, and B is U_i or V minus the respective point.

The two isomorphisms from the long exact sequence can be computed similar to **Corollary 2.8.40**. Briefly, we have

$$H_n(U_i) \to H_n(U_i, U_i \setminus \{x_i\}) \to H_{n-1}(U_i \setminus \{x_i\}) \to H_{n-1}(U_i).$$

We recognize $H_n(U_i) = H_{n-1}(U_i) = 0$ and $U_i \setminus \{x_i\} \simeq S^{n-1}$, so $H_n(U_i, U_i \setminus \{x_i\}) \cong H_{n-1}(S^{n-1})$.

These isomorphisms cause the triangles and squares to commute. From this we can conclude the top homomorphism $f_* : H_n(U_i, U_i \setminus \{x_i\}) \to H_n(V, V \setminus \{y\})$ ($f_* : \mathbf{Z} \to \mathbf{Z}$) is multiplication by an integer; i.e., $f(\alpha) = d\alpha$.

Definition 2.8.61. We call this top map $f_* : H_n(U_i, U_i \setminus \{x_i\}) \to H_n(V, V \setminus \{y\})$, i.e., $f : \mathbf{Z} \to \mathbf{Z}$, defined by $f(\alpha) = d\alpha$, the **local degree of f at x_i**. We write $\deg(f)|_{x_i}$.

Theorem 2.8.62. $\deg(f) = \displaystyle\sum_i \deg(f)|_{x_i}$.

Proof. Since $H_n(S^n, S^n \setminus f^{-1}(y)) = H_n(\cup U_i, \cup U_i \setminus \{x_i\})$ by excision (let $X = S^n$, $A = S^n \setminus f^{-1}(y)$, and $B = S^n \setminus \cup U_i$), and $H_n H_n(\cup U_i, \cup U_i \setminus \{x_i\}) = \oplus H_n(U_i, U_i \setminus \{x_i\})$, and k_i in the diagram above is just inclusion of the ith summand, we have $p_i j(1) = 1$. Thus $j(1) = (1, 1, ..., 1) = \sum k_i(1)$. Therefore considering the middle map $f_* : H_n(S^n, S^n \setminus f^{-1}(y)) \to H_n(S^n, S^n \setminus \{y\})$, we have $f_*(\sum k_i(1)) = \sum \deg(f)|_{x_i}$. Therefore, considering the bottom f_* map, $\deg(f) = f_*(1) = \sum \deg(f)|_{x_i}$, as claimed. $\square$

Proposition 2.8.63. *If $f : \boldsymbol{C} \to \boldsymbol{C}$ is a polynomial, there is a map $\hat{f} : \hat{\boldsymbol{C}} \to \hat{\boldsymbol{C}}$ where $\hat{\boldsymbol{C}}$ is the Riemann sphere (i.e., we may think of $\hat{f} : S^2 \to S^2$.*

Then $\deg(\hat{f}) = \deg(f)$ (typical polynomial degree), and if $f(z) = 0$, then $\deg(\hat{f})\Big|_z$ is the multiplicity of the root.

Example 2.8.64. Let $f : S^1 \to S^1$ be $z \mapsto z^2$. Then we see $\deg(f) = 2$. Let's also show it.

The quotient map $\sigma : I \to {}^{I}\!/_{\partial I} = S^1$ generates $H_1(S^1) = \mathbf{Z}$. Thus $f\sigma$ wraps the interval I twice around S^1. We need to show that $f\sigma$ is homologous to 2σ as singular cycles.

To see this, there exists a singular 2-chain $\xi = \sum n_\alpha \sigma_\alpha^2$ with $\partial\xi = f\sigma - 2\sigma$. It's a pair of pants.

Lemma 2.8.65. *If X is a CW-complex, then*

1. *$H_k(X^n, X^{n-1})$ is zero when $n \neq k$ and free abelian with basis the n-cells of X when $n = k$,*
2. *$H_k(X^n) = 0$ for $k > n$, and*
3. *the inclusion $\iota : X^n \hookrightarrow X$ induces an isomorphism on homology $\iota_* : H_k(X^n) \xrightarrow{\sim} H_k(X)$ for $k < n$.*

Proof. Let X be a CW-complex.

1. By excision,$H_k(X^n, X^{n-1}) \cong \widetilde{H_k}\left({}^{X^n}\!/_{X^{n-1}}\right)$. Then $\widetilde{H_k}\left({}^{X^n}\!/_{X^{n-1}}\right) \cong H_k\left(\bigvee_\alpha S^n{}_\alpha\right)$, where the wedge is indexed by the n-cells in X. But $H_k\left(\bigvee_\alpha S^n{}_\alpha\right) = \bigoplus_\alpha \mathbf{Z}_\alpha$, as claimed.
2. If $k > n$, then we have the excision exact sequence
$$\cdots \to H_k(X^{n-1}) \to H_k(X^n) \to H_k(X^n, X^{n-1}) \to \cdots$$
Since $k > n$, we know that $H_k(X^{n-1}) = 0$ via induction, and we have $H_k(X^n, X^{n-1}) = 0$ by above. So we have $0 \to H_k(X) \to 0$, so $H_k(X) = 0$.
3. If X is finite dimensional, then if $k \neq n$ and $k \neq n+1$, then
$$0 = H_{k+1}(X^{n+1}, X^n) \to H_k(X^n) \to H_k(X^{n+1}) \to H_k(X^{n+1}, X^n) = 0$$
Thus if $k < n$, $H_k(X^n) \cong H_k(X^{n+1})$ since $k \neq n, n{+}1$. Then $H_k(X^{n+1}) \cong H_k(X^{n+2})$ since $k \neq n+1, n+2$. Continue in this way, and we conclude that $H_k(X^n) \cong ... \cong H_k(X^{n+m}) = H_k(X)$ for some m.

   If, however, X is not finite dimensional, then any singular chain has image contained in a finite skeleton. Then we can string together protions of the long exact sequences for (X^{n+1}, X^n), (X^n, X^{n-1}), and (X^{n-1}, X^{n-2}) into the following diagram:

$$
\begin{tikzcd}
 & & \vdots \arrow[d] & & & \\
 & & H_{n+1}(X^{n+1}, X^n) \arrow[dr, "\partial_{n+1}"] \arrow[dd, "d_{n+1}"] & & 0 = H_n(X^{n-1}) \arrow[dl] & \\
 & & & H_n(X^n) \arrow[dl, "j_n"] \arrow[dr] & & \\
H_{n-1}(X^{n-2}) = 0 \arrow[dr] & & H_n(X^n, X^{n-1}) \arrow[dl, "\partial_n"] \arrow[dd, "d_n"] & & H_n(X^{n+1}) \cong H_n(X) \arrow[dr] & \\
 & H_{n-1}(X^{n-1}) \arrow[dr, "j_{n-1}"] & & & & 0 = H_n(X^{n+1}, X^n) \\
 & & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[d] & & & \\
 & & \vdots & & &
\end{tikzcd}
$$

Since the diagram commutes, $d_n = j_{n-1}\partial_n$. We claim that d_n is a boundary homomorphism, i.e., that $d_n d_{n+1} = 0$. To see this, we have $d_n d_{n+1} = j_{n-1}\partial_n j_n \partial_{n+1} = j_{n-1} 0 \partial_{n+1} = 0$, since $\partial_n j_n$ is 0 because it is two successive maps from the long exact sequence for (X^n, X^{n-1}). $\square$

Thus, we have a chain complex, and we can define:

Definition 2.8.66. We define **cellular homology** to be the homology on the chain complex whose chain groups are $H_n(X^n, X^{n-1})$ and whose boundary maps are $d_n : H_n(X^n, X^{n-1}) \to H_{n-1}(X^{n-1}, X^{n-2})$ as above. Thus we write

$$H_n^{CW}(X) = {}^{\ker d_n}\!/_{\operatorname{Im} d_{n+1}}.$$

Theorem 2.8.67. *Our new homology theory agrees with the others; i.e., we have that $H_n^{CW}(X) \cong H_n(X)$.*

Proof. We have $H_n(X) = {}^{H_n(X^n)}\!/_{\operatorname{Im} \partial_{n+1}}$ since, in the above strung-together diagram, the long exact sequence string that runs down and right is:

$$H_{n+1}(X^{n+1}, X^n) \xrightarrow{\partial_{n+1}} H_n(X^n) \to H_n(X) \to 0.$$

Now ${}^{H_n(X^n)}\!/_{\operatorname{Im} \partial_{n+1}}$ is exactly elements of $H_n(X^n)$ that are not in $\operatorname{Im} \partial_{n+1}$, which, since the sequence is exact, is exactly the elements that get sent to $H_n(X)$, since elements in $\operatorname{Im} \partial_{n+1}$ get sent to 0. Thus we do have $H_n(X) = {}^{H_n(X^n)}\!/_{\operatorname{Im} \partial_{n+1}}$.

Now, again referring to the diagram above, as $\ker j_n = \operatorname{Im}(0 \to H_n(X^n)) = 0$, $\operatorname{Im} \partial_{n+1}$ is mapped surjectively onto $\operatorname{Im}(j_n \partial_{n+1}) = \operatorname{Im} d_{n+1}$, and $H_n(X^n)$ is mapped isomorphically onto $\operatorname{Im} j_n = \ker \partial_n$.

And since $\ker j_{n-1} = 0$, $\ker \partial_n = \ker d_n$.

Therefore, j_n induces an isomorphism ${}^{H_n(X^n)}\!/_{\operatorname{Im} \partial_{n+1}} \xrightarrow{\sim} {}^{\ker d_n}\!/_{\operatorname{Im} d_{n+1}}$, and the homology theories are isomorphic. $\square$

Note the following factorization:

$$
\begin{tikzcd}[column sep=small]
n\text{-cells} \arrow[rr, mapsto] & & (n-1)\text{-cells} \\
H_n(X^n, X^{n-1}) \arrow[rr, "d_n"] \arrow[dr, "\partial"'] & & H_n(X^{n-1}, X^{n-2}) \\
& H_{n-1}(X^{n-1}) \arrow[ur, "j"'] &
\end{tikzcd}
$$

Example 2.8.68. We claim a CW structure on $\mathbf{CP}^n = {S^{2n+1}}/{z \sim \lambda z}$ where $|\lambda| = 1$, $\lambda \in \mathbf{C}$ is $e^0 \cup e^2 \cup ... \cup e^{2n}$.

Therefore, the cellular chain complex is

$$\cdots \to 0 \to 0 \to \underset{(\dim 2n)}{\mathbf{Z}} \to 0 \to \underset{(\dim 2n-2)}{\mathbf{Z}} \to \cdots \to \underset{(\dim 2)}{\mathbf{Z}} \to 0 \to \underset{(\dim 0)}{\mathbf{Z}} \to 0$$

Thus every homomorphism d_k must be the zero map, so

$$H_k(\mathbf{CP}^n) = {\ker d_k}/{\operatorname{Im} d_{k+1}} = \begin{cases} \mathbf{Z} & \text{if } k = 0, 2, ..., 2n, \\ 0 & \qquad \text{otherwise.} \end{cases}$$

Let's explore the boundary maps d_n of cellular homology more explicitly. We'd like to be able to compute them.

First, for $d_0 : H_0(X^0, \emptyset) = H_0(X^0) \cong C_0(X^0) \to 0$; we see this must be the zero map.

Next, for $d_1 : H_1(X^1, X^0) \to H_0(X^0, \emptyset) = H_0(X^0) = C_0(X^0)$, this is the same as the simplicial boundary map $\Delta_1(X) \xrightarrow{\partial_1} \Delta_0(X)$.

Now, for $n > 1$, we claim that $d_n(e^n{}_\alpha) = \sum_\beta d_{\alpha\beta} e^{n-1}{}_\beta$, where $d_{\alpha\beta}$ is the degree of the map $\Delta_{\alpha\beta} : S^{n-1}{}_\alpha \to S^{n-1}{}_\beta$, as set up below:

Let $\partial D^n{}_\alpha = S^{n-1}{}_\alpha \xrightarrow{\varphi_\alpha} X^{n-1}$ where $\varphi_\alpha : \partial D^n{}_\alpha \to X^{n-1}$ be the attaching map and let $\Phi_\alpha : D^n{}_\alpha \to X$ be the characteristic map.

Then we have

$$\Delta_{\alpha\beta} : S^{n-1}{}_\alpha \xrightarrow{\varphi_\alpha} X^{n-1} \xrightarrow{q} {X^{n-1}}/{X^{n-2}} \xrightarrow{q\beta} {X^{n-1}}/{X^{n-1} \setminus e^{n-1}{}_\beta} = S^{n-1}{}_\beta.$$

We remark that q is the obvious quotient map, that ${X^{n-1}}/{X^{n-2}}$ is just a wedge of spheres, and that $q\beta$ is the collapse of the complement of a cell $e^{n-1}{}_\beta$ to a point.

Thus the composition $\Delta_{\alpha\beta}$ is the map such that $\deg(\Delta_{\alpha\beta}) = d_{\alpha\beta}$.

To see this claim, consider the commutative diagram:

$$
\begin{tikzcd}
{[D^n{}_\alpha]} \arrow[d, mapsto] & H_n(D^n{}_\alpha, \partial D^n{}_\alpha) \arrow[r, "\partial"] \arrow[d, "(\Phi_\alpha)_*"] & \widetilde{H_{n-1}}(\partial D^n{}_\alpha) \arrow[r, "(\Delta_{\alpha\beta})_*"] \arrow[d, "(\varphi_\alpha)_*"] & \widetilde{H_{n-1}}(S^{n-1}{}_\beta) \\
e^n{}_\alpha & H_n(X^n, X^{n-1}) \arrow[r, "\partial"] \arrow[dr, "d_n"'] & \widetilde{H_{n-1}}(X^{n-1}) \arrow[r, "q_*"] \arrow[d, "j"] & \widetilde{H}\left({X^{n-1}}/{X^{n-2}}\right) \arrow[u, "(q\beta)_*"'] \\
& & H_n(X^{n-1}, X^{n-2}) \arrow[ur, "\sim"'] &
\end{tikzcd}
$$

The map $H_n(X^{n-1}, X^{n-2}) \xrightarrow{\sim} \widetilde{H}\left(X^{n-1}\big/X^{n-2}\right)$ is an isomorphism by excision.

The map $(q\beta)_*$ is just projection onto the βth summand.

Then $d_n(e^n{}_\alpha) = j(\varphi_\alpha)_*\partial[D^n{}_\alpha] = j(\varphi_\alpha)_*[\partial D^n{}_\alpha]$, and so $(q\beta)_* d_n(e^n{}_\alpha) = (\Delta_{\alpha\beta})_*[\partial D^n{}_\alpha]$.

Now, how do we actually use this to compute cellular homology?

Example 2.8.69. Let S_g be the orientable surface of genus g. Then we have

$$\cdots \to 0 \to \underset{\text{one 2-cell}}{\mathbf{Z}} \xrightarrow{d_2} \underset{2g\text{ 1-cells}}{\mathbf{Z}^{2g}} \xrightarrow{d_1} \underset{\text{one 0-cell}}{\mathbf{Z}} \xrightarrow{d_0} 0.$$

Then d_0 is the zero map. Also, d_1 is the zero map, since there is a single 0-cell. Finally d_2 is also zero, since the map $\partial D^2 \to X^1 = \bigvee_{i=1}^{2g} S^1{}_i$ crosses each $S^1{}_i$ factor with local degrees 1 and -1, so the total degree is 0.

Therefore, since all the maps are the zero map, we have seen that the homology is the same as the chain complex:

$$H_n(S_g) = \begin{cases} \mathbf{Z} & \text{if } n = 0, 2, \\ \mathbf{Z}^{2g} & \text{if } n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Example 2.8.70. Let N_g be the nonorientable surface $\mathop{\#}_g \mathbf{RP}^2$. Then we have the chain complex

$$\cdots \to 0 \to \underset{\text{one 2-cell}}{\mathbf{Z}} \xrightarrow{d_2} \underset{g\text{ 1-cells}}{\mathbf{Z}^{g}} \xrightarrow{d_1} \underset{\text{one 0-cell}}{\mathbf{Z}} \xrightarrow{d_0} 0.$$

Then d_0 is the zero map, d_1 is the zero map since there is a single 0-cell, and we see that $d_2(1) = (2, 2, ..., 2)$, since each $S^1{}_i$ factor is mapped on with local degrees 1 and 1, so the total degree is 2. Therefore, d_2 is injective.

Now, since $(1, 0, ..., 0, 0), (0, 1, ..., 0, 0), (0, 0, ..., 1, 0), (1, 1, ..., 1, 1)$ is a basis for $\mathbf{Z}^g$, we see that

$$H_1(N_g) = {\ker d_1}\big/{\operatorname{Im} d_2} = \mathbf{Z}^g\big/\langle\langle(2, ..., 2)\rangle\rangle = \mathbf{Z}^{g-1} \oplus \mathbf{Z}_2.$$

Therefore,

$$H_n(N_g) = \begin{cases} \mathbf{Z} & \text{if } n = 0, \\ \mathbf{Z}^{g-1} \oplus \mathbf{Z}_2 & \text{if } n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Example 2.8.71. Let $X = S^1{}_a \vee S^1{}_b \bigcup_f D^2{}_1 \amalg D^2{}_2$, where the attaching f takes the boundary of $D^2{}_1$ to $a^5 b^{-3}$ and the boundary of $D^2{}_2$ to $b^3(ab)^{-2}$.

Then the map $d_2 : \mathbf{Z}^2 \to \mathbf{Z}^2$ taking 2-cells to 1-cells is given by the matrix

$$A = \begin{bmatrix} 5 & -2 \\ -3 & 1 \end{bmatrix}$$

and $\det(A) = -1$, which implies that A is invertible over $\mathbf{Z}$. Thus d_2 is an isomorphism. This implies $\ker d_2 = 0$, so $H_2(X) = 0$, and $\ker d_1 = \operatorname{Im} d_2$, so $H_1(X) = 0$.

Thus we can conclude that $\widetilde{H_i}(X) = 0$ for all i (using reduced since $H_0(X) = \mathbf{Z}$). Such a space X is called acyclic when its reduced homology vanishes.

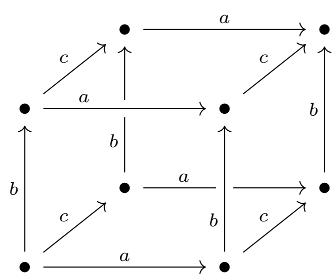
We know another acyclic space is a point, so homology cannot tell the difference between this space and a point. However, this is not a point; X is neither contractible nor simply connected!

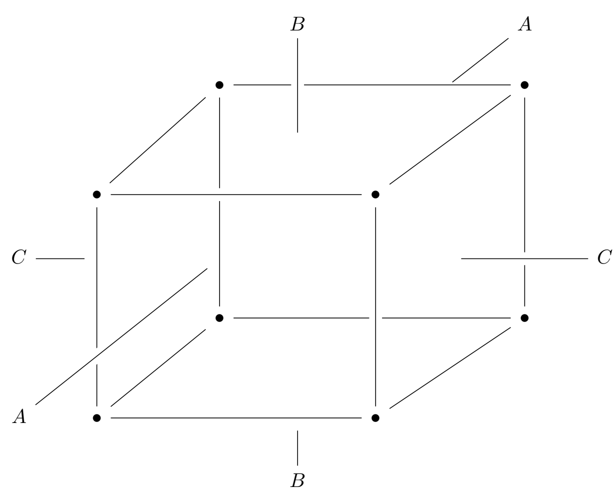
To see this, we can use **Van Kampen's Theorem 2.4.3** to get $\pi_1(X) = \langle a, b \mid a^5 b^{-3}, b^3(ab)^{-2}\rangle$. Now let $G = A_5$ be the group of orientation preserving symmetries of a regular dodecahedron (D12). Then let $\rho_2 = \frac{2\pi}{5}$ correspond to rotation through the center of a face, i.e., $(12345) \in A_5$. Also $\rho_b = \frac{2\pi}{3}$ is rotation around a vertex, $(254) \in A_5$. Then $\rho_a \rho_b$ is rotation by π through the midpoint of an edge, $(12)(34) \in A_5$. Therefore, the relations $id = (\rho_a)^5 = (\rho_b)^3 = (\rho_a\rho_b)^2$ imply that there is a homomorphism $f : \pi_1(X) \to G$, sending $a \mapsto \rho_a$ and $b \mapsto \rho_b$, which is surjective, since ρ_a and ρ_b generate G.

But it can be seen that $\pi_1(X)$ is not the trivial group! In fact, $\ker f \cong \mathbf{Z}_2$. Thus $|\pi_1(X)| = 120$ as $|A_5| = 60$.

This space is called the Poincaré dodecahedral space.

Example 2.8.72. Let $X = T^3 = S^1 \times S^1 \times S^1$. See the two pictures, one which shows how to identify edges and one showing how to identify faces:

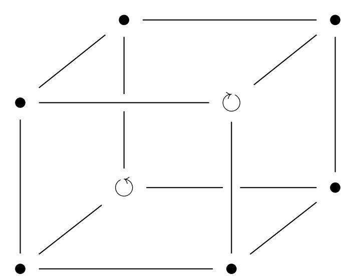




Then we have one 3-cell, three 2-cells A, B, C, three 1-cells a, b, c, and one 0-cell, the vertex. The cellular chain complex is therefore

$$0 \to \mathbf{Z} \xrightarrow{d_3} \mathbf{Z}^3 \xrightarrow{d_2} \mathbf{Z}^3 \xrightarrow{d_1} \mathbf{Z} \to 0$$

Then d_1 is the zero map, because there is a single 0-cell, d_2 is the zero map, because in each face there are the same number of edges oriented positively as negatively, and d_3 is the zero map, because there are three maps $\Delta_{\alpha\beta} : \partial D^3 = S^2 \to S^2 = {D^2}_\beta / {\partial D^2}_\beta$ that differ by a reflection; each of the three maps has local degrees 1 and -1 as two pieces differ by reflection.



Since the d_ks are all the zero map,

$$H_n(T^3) = \begin{cases} \mathbf{Z} & \text{if } n = 0, 3, \\ \mathbf{Z}^3 & \text{if } n = 1, 2 \\ 0 & \text{otherwise.} \end{cases}$$

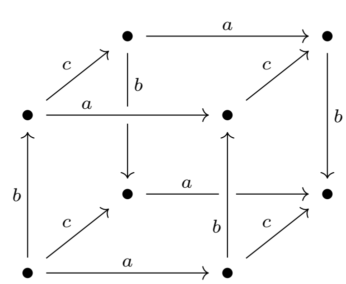
As a quick aside, check out the awesome Pascal's triangle relationship:

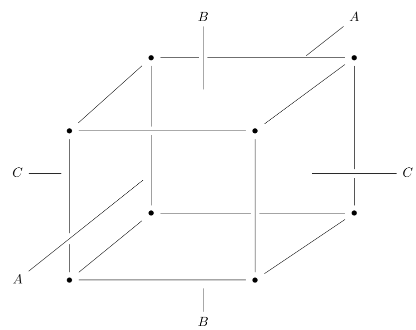
$$T^1 = S^1 : \quad 0 \longrightarrow \mathbf{Z} \longrightarrow 0$$

$$T^2 = S^1 \times S^1 : \quad 0 \longrightarrow \mathbf{Z} \longrightarrow \mathbf{Z}^2 \longrightarrow \mathbf{Z} \longrightarrow 0$$

$$T^3 = S^1 \times S^1 \times S^1 : \quad 0 \longrightarrow \mathbf{Z} \longrightarrow \mathbf{Z}^3 \longrightarrow \mathbf{Z}^3 \longrightarrow \mathbf{Z} \longrightarrow 0$$

Example 2.8.73. Now let K be the Klein bottle and consider the space $K \times S^1$. We give the two pictures again:





The cellular chain complex is as before;

$$0 \to \mathbf{Z} \xrightarrow{d_3} \mathbf{Z}^3 \xrightarrow{d_2} \mathbf{Z}^3 \xrightarrow{d_1} \mathbf{Z} \to 0$$

We have d_1 is the zero map because there is a single 0-cell. Then see that $d_2(A) = 0$, $d_2(B) = 0$, but $d_2(C) = 2b$. Finally, $d_3(e^3) = 2A$, since that face differs by rotation, not reflection. Therefore, we have

$$H_n(K \times S^1) = \begin{cases} \mathbf{Z} & \text{if } n = 0, \\ \ker d_1 / \operatorname{Im} d_2 = \mathbf{Z}^3 / \langle\langle 2b \rangle\rangle = \mathbf{Z}^2 \oplus \mathbf{Z}_2 & \text{if } n = 1, \\ \ker d_2 / \operatorname{Im} d_3 = \langle\langle A, B \rangle\rangle / \langle\langle 2A \rangle\rangle = \mathbf{Z} \oplus \mathbf{Z}_2 & \text{if } n = 2, \\ 0 & \text{otherwise.} \end{cases}$$

2.9 Section 9: The Euler Characteristic

Definitions: Euler characteristic
Main Idea: We define the Euler characteristic and show that it is a topological invariant.

Definition 2.9.1. If we have a finite dimensional CW-complex X, then we define the **Euler characteristic** of X, $\chi(X)$, to be the alternating sum over n of the number of n-cells in X; i.e., if c_n is the number of n-cells of X, then

$$\chi(X) = \sum_n (-1)^n c_n.$$

It's not immediately clear that this is independent of the topology on X. However, we know that homology is topologically invariant by **Theorem 2.8.25**, so the following theorem does it for us:

Theorem 2.9.2.

$$\chi(X) = \sum_n (-1)^n \operatorname{rank} H_n(X).$$

Proof. Consider the cellular chain complex

$$0 \to C_k \xrightarrow{d_k} C_{k-1} \to ... \xrightarrow{d_1} C_0 \to 0.$$

We get short exact sequences

$$0 \to \ker d_n \to C_n \to \operatorname{Im} d_n \to 0;$$

thus $\operatorname{rank} C_n = \operatorname{rank}\ker d_n + \operatorname{rank}\operatorname{Im} d_n$. (This is just the **Rank-Nullity Theorem**.)

Now, as we have a chain complex,

$$0 \to \operatorname{Im} d_{n+1} \to \ker d_n \to H_n(X) \to 0$$

is exact as well, so $\operatorname{rank}\ker d_n = \operatorname{rank} H_n(X) + \operatorname{rank}\operatorname{Im} d_{n+1}$.

Therefore

$$\begin{aligned}\sum_n (-1)^n \operatorname{rank} C_n &= \sum_n (-1)^n \left(\operatorname{rank} H_n(X) + \operatorname{rank}\operatorname{Im} d_{n+1} + \operatorname{rank}\operatorname{Im} d_n\right) \\ &= \sum_n (-1) \operatorname{rank} H_n(X)\end{aligned}$$

via telescoping. Thus we have it. □

Example 2.9.3. Since S_g has one 0-cell, $2g$ 1-cells, and one 2-cell, $\chi(S_g) = 1 - 2g + 1 = 2 - 2g$.

Example 2.9.4. N_g has a single 0-cell, g 1-cells, and one 2-cell. Thus $\chi(N_g) = 1 - g + 1 = 2 - g$.

2.10 Section 10: Homology II: Mayer-Vietoris, Homology with Coefficients

Definitions: Heegaard splitting, homology with coefficients
Main Idea: Mayer-Vietoris computes homology by decomposing your space into nice subspaces, just like Van Kampen for the fundamental group. Homology with coefficients serves to give more information if using $\mathbf{Z}$ isn't enough; i.e., using $\mathbf{Z}_2$ for nonorientable spaces.

Mayer-Vietoris is the Van Kampen analogue for homology.

Theorem 2.10.1 (Mayer-Vietoris). *If $X = \operatorname{Int}(A) \cup \operatorname{Int}(B)$, then the Mayer-Vietoris sequence,*

$$\begin{tikzcd} \cdots \arrow[r] & H_n(A \cap B) \arrow[r] & H_n(A) \oplus H_n(B) \arrow[r] & H_n(X) \arrow[dll, "\partial"] \\ & H_{n-1}(A \cap B) \arrow[r] & H_{n-1}(A) \oplus H_{n-1}(B) \arrow[r] & \cdots \end{tikzcd}$$

is exact.

Proof. We're going to be short and sweet in this proof. It follows from the short exact sequence of chain complexes:

$$0 \to C_n(A \cap B) \xrightarrow{\varphi} C_n(A) \oplus C_n(B) \xrightarrow{\psi} C_n{}^U(X) \to 0$$

where $U = \{A, B\}$, so $C_n{}^U$ are chains in X of the form $\xi_A + \xi_B$ for $\xi_A \in C_n(A)$ and $\xi_B \in C_n(B)$.

Now, $\varphi(x) = (x, -x)$ and $\psi(x, y) = x + y$ clearly is exact.

Earlier in the proof of **Excision 2.8.49**, we saw $C_n{}^U \hookrightarrow C_n(X)$ induces an isomorphism on homology.

The boundary map is $\partial : H_n(X) \to H_{n-1}(A \cap B)$. To define it, represent an element $\alpha \in H_n(X)$ by a cycle $x + y \in C_n{}^U(X)$, where $x \in A$, $y \in B$. Then we do the **Snake Lemma** again:

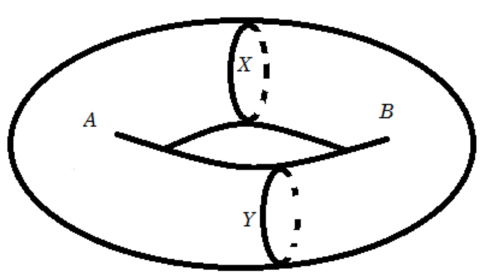
$$\begin{tikzcd}
& & \partial x \arrow[d, mapsto] & C_{n-1}(A \cap B) \arrow[dd] \\
(x,y) \arrow[rr, mapsto] \arrow[dd, mapsto] & & (\partial x, \partial y) & \\
& C_n(A) \oplus C_n(B) \arrow[rr, "\partial"] \arrow[d, "j"] & & C_{n-1}(A) \oplus C_{n-1}(B) \\
x + y & C_n{}^U(X) & &
\end{tikzcd}$$

Now, x and y might not be cycles, but $\partial x = -\partial y$, so $\partial\alpha = [\partial x] \in H_{n-1}(A \cap B)$, well-defined as last time. $\square$

Note that if $X = A \cup B$ and A and B are deformation retracts of open sets, then **Mayer-Vietoris 2.10.1** still holds; A and B need not be open sets themselves.

Example 2.10.2. Let $S^n = D^n{}_+ \cup D^n{}_-$. Then using Mayer-Vietoris, since $\widetilde{H_k}(D^n{}_+) \oplus \widetilde{H_k}(D^n{}_-) = 0$, we have $\widetilde{H_k}(S^n) \cong \widetilde{H_{k-1}}(S^{n-1})$.

Example 2.10.3. Let $S^1 \times S^1 = A \cup B$ where we cut the torus in half through its hole with cuts X and Y and get two cylinders, A and B. See the picture:



Then $\widetilde{H}_k(A) = \widetilde{H}_k(B) = \widetilde{H}_k(S^1) = \mathbf{Z}$ if $k = 1$ and 0 otherwise. And $\widetilde{H}_k(A \cap B) = \widetilde{H}_k(X \amalg Y) = \widetilde{H}_k(S^1 \amalg S^1)$ is $\mathbf{Z}$ when $k = 0$, $\mathbf{Z}^2$ when $k = 1$, and 0 otherwise. Thus we have

$$\begin{tikzcd}
& 0 \arrow[r] & \widetilde{H}_2(S^1 \times S^1) \arrow[dll, out=-20, in=160] \\
\mathbf{Z}^2 \arrow[r, "\varphi_*"] & \mathbf{Z}^2 \arrow[r, "\psi_*"] & \widetilde{H}_1(S^1 \times S^1) \arrow[dll, out=-20, in=160] \\
\mathbf{Z} \arrow[r] & 0 &
\end{tikzcd}$$

We have $\varphi_*(x, y) = (x - y, y - x)$, so $\ker \varphi_* \cong \mathbf{Z}$. Therefore, $H_2(S^1 \times S^1) \cong \mathbf{Z}$, as we have seen before.

We get a short exact sequence

$$0 \to \mathbf{Z}^2 / \operatorname{Im} \varphi_* = \mathbf{Z} \to \widetilde{H}_1(S^1 \times S^1) \to \mathbf{Z} \to 0.$$

Since everything is free abelian, the sequence splits, and we have $\widetilde{H}_1(S^1 \times S^1) = \mathbf{Z} \oplus \mathbf{Z} = \mathbf{Z}^2$.

In the last example, we said that the sequence splits. What specifically does this mean?

Lemma 2.10.4 (**Splitting Lemma**). *If we have a short exact sequence of abelian groups*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "i"] & B \arrow[r, "j"] \arrow[l, bend left, dotted, "p"] & C \arrow[r] \arrow[l, bend left, dotted, "s"] & 0
\end{tikzcd}$$

then the following are equivalent:

1. *"The sequence splits."*
2. *There exists $p : B \to A$ such that $pi = id_A$.*
3. *There exists $s : C \to B$ such that $js = id_C$.*
4. *There exists an isomorphism such that $B \cong A \oplus C$ and the following diagram commutes.*

$$\begin{tikzcd}
& & B \arrow[dd, "\wr"] \arrow[dr, "j"] & & \\
0 \arrow[r] & A \arrow[ur, "i"] \arrow[dr] & & C \arrow[r] & 0 \\
& & A \oplus C \arrow[ur] & &
\end{tikzcd}$$

Note that if C is free, then $s : C \to B$ always exists; simply fix a basis $\{c_\alpha\}$ for C and choose $b_\alpha \in j^{-1}(c_\alpha)$, and set $s(c_\alpha) = b_\alpha$.

Proof. We only sketch the proof; while we will make good use of the **Splitting Lemma**, it is purely an algebraic statement.

For 2. implies 4., we construct the isomorphism: $B \to A \oplus C$ via $b \mapsto (p(b), j(b))$. This map is surjective, as given $(a, c) \in A \oplus C$, choose $b' \in j^{-1}(c)$. Let $b = i(a - p(b')) + b'$. Applying j is good. Next, $p(b) = pi(a - p(b')) + p(b') = a - p(b') + p(b') = a$ and for j, $j(b) = ji(a - p(b')) + j(b') = 0 + c = c$. We can also see that this map is surjective; follow your nose.

For 3. implies 4., the map is $A \oplus C \to B$, $(a, c) \mapsto i(a) + s(c)$.

For 4. implies 2. and 3., the proof is obvious. If $B \cong A \oplus C$, we know how to project or include. $\square$

Corollary 2.10.5. *Given $A \subseteq X$, if there exists a retraction $r : X \to A$, then $r_* \iota_* = id_{H_n(A)}$, so ι_* is injective. Then the long exact sequence*

$$H_n(A) \xrightarrow{\iota_*} H_n(X) \to H_n(X, A)$$

breaks into short exact sequences:

$$0 \to H_n(A) \xrightarrow{\iota_*} H_n(X) \to H_n(X, A) \to 0.$$

Furthermore, we have the projection

$$\begin{tikzcd} 0 \arrow[r] & H_n(A) \arrow[r, "\iota_*"] & H_n(X) \arrow[r] \arrow[l, dotted, bend left, "r_*"] & H_n(X, A) \arrow[r] & 0, \end{tikzcd}$$

and by the ***Splitting Lemma 2.10.4****, $H_n(X) = H_n(A) \oplus H_n(X, A)$.*

Note that this can be used to show the nonexistence of retractions, like **Brouwer's Fixed Point Theorem 1.55.6 2.8.41**.

Example 2.10.6. The Möbius band X does not retract onto its boundary S^1.

If it did, we have short exact sequences

$$0 \to H_1(S^1) = \mathbf{Z} \xrightarrow{2} H_1(X) = \mathbf{Z} \to H_1(X, S^1) = \mathbf{Z}_2 \to 0,$$

but this sequence does not split, since $\mathbf{Z} \neq \mathbf{Z} \oplus \mathbf{Z}_2$.

Example 2.10.7. Let A and B be genus g handlebodies; i.e., thickened graphs with $\pi_1(A) = \pi_1(B) = \mathbf{F}_g$. Equivalently, A and B are surfaces S_g with their insides. Then let $X = A \sqcup_f B$ where $f : \partial A \to \partial B$ (i.e., $f : S_g \to S_g$) is a homeomorphism. Now we use Mayer-Vietoris to compute the homology groups of X.

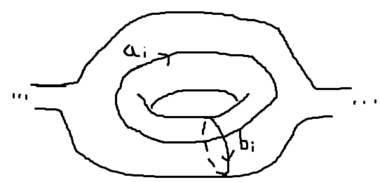
Since A and B deformation retract onto wedges of circles,

$$\widetilde{H}_k(A) = \widetilde{H}_k(B) = \begin{cases} \mathbf{Z}^g & \text{if } k = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, since $A \cap B = S_g$, (recall **Example 2.8.69**),

$$\widetilde{H}_k(A \cap B) = \begin{cases} 0 & \text{if } k = 0, \\ \mathbf{Z}^{2g} & \text{if } k = 1, \\ \mathbf{Z} & \text{if } k = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Now let's look at the basis for the homologies. If a_i and b_i are as pictured, then a_i and b_i are generators for the first homology of the surface S_g.



In other words, $\langle a_i, b_i \rangle_{i=1}^{g} \cong \mathbf{Z}^{2g} = \widetilde{H}_1(\partial A)$.

However, in the handlebody, b_i are all nulhomologous, since they are the boundaries of a disk. Thus $\widetilde{H}_1(A) = \langle a_i \rangle_{i=1}^{g}$.

Then, $\iota_A : H_1(\partial A) \to H_1(A)$ is projection onto the first g coordinates. Likewise, $\iota_B : H_1(\partial B) \to H_1(B)$. Thus, the relevant part of the Mayer-Vietoris sequence is:

$$\begin{tikzcd}
 & 0 \arrow[r] & \widetilde{H}_3(X) \arrow[dll, out=-30, in=150] & \\
\widetilde{H}_2(A \cap B) \arrow[r] & 0 \arrow[r] & \widetilde{H}_2(X) \arrow[dll, out=-30, in=150] & \\
\widetilde{H}_1(A \cap B) \arrow[r, "\varphi"] & \widetilde{H}_1(A) \oplus \widetilde{H}_1(B) \arrow[r, "\psi"] & \widetilde{H}_1(X) \arrow[r] & 0
\end{tikzcd}$$

So $\varphi(\alpha) = (\iota_A(\alpha), -\iota_B f_*(\alpha))$, where f_* is the induced map $f_* : \widetilde{H}_1(\partial A) \to \widetilde{H}_1(\partial B)$ (i.e., $f_* : \mathbf{Z}^{2g} \to \mathbf{Z}^{2g}$). And $\psi(a, b) = a + b$.

By exactness of the Mayer-Vietoris sequence, $\widetilde{H}_2(X) \cong \ker \varphi = \{\alpha \in \widetilde{H}_1(A \cap B) \mid \iota_A(\alpha) = 0, -\iota_B f_*(\alpha) = 0\}$.

We can consider f_* as a $2g \times 2g$ matrix with $g \times g$ submatrices:

$$f_*(\alpha) = \begin{bmatrix} M_1 & N_1 \\ M_2 & N_2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} M_1\alpha_1 + N_1\alpha_2 \\ M_2\alpha_1 + N_2\alpha_2 \end{bmatrix},$$

where α_1 and α_2 correspond to the a_is and b_is.

Therefore $\ker\varphi = \{(\alpha_1, \alpha_2) \in H_1(A\cap B) \mid \alpha_1 = 0 \text{ and } N_1\alpha_2 = 0\} = \ker N_1 \subseteq \mathbf{Z}^g$.

For the first homology,

$$\widetilde{H_1}(X) \cong {\widetilde{H_1}(A) \oplus \widetilde{H_1}(B)}\big/{\ker\psi} \cong {\widetilde{H_1}(A) \oplus \widetilde{H_1}(B)}\big/{\operatorname{Im}\varphi}$$

by exactness.

Now,

$$\begin{aligned}\operatorname{Im}\varphi &= \left\{(\alpha_1, M_1\alpha_1 + N_1\alpha_2) \in \widetilde{H_1}(A) \oplus \widetilde{H_1}(B) \mid \begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix} \in \widetilde{H_1}(A\cap B)\right\}\\ &\cong \operatorname{Im}\begin{bmatrix} I_g & 0\\ M_1 & N_1\end{bmatrix}\\ &\subseteq \mathbf{Z}^{2g}.\end{aligned}$$

Thus $\operatorname{rank} H_1(X) = 2g - \operatorname{rank}\begin{bmatrix} I_g & 0\\ M_1 & N_1\end{bmatrix} = g - \operatorname{rank} N_1$, and $\operatorname{rank} H_2(X) = \operatorname{rank}\ker N_1 = g - \operatorname{rank} N_1$. These are the same![18]

Note that this says the following about the Euler characteristic:

$$\begin{aligned}\chi(X) &= \sum_i (-1)^i \operatorname{rank} H_i(X)\\ &= 1 - (g - \operatorname{rank} N_1) + (g - \operatorname{rank} N_1) - 1\\ &= 0.\end{aligned}$$

Therefore:

Theorem 2.10.8. *Any three-manifold X obtained by gluing together two handlebodies, i.e., $X = A \sqcup_f B$ for some f, is such that $\chi(X) = 0$.*

Proof. **Example 2.10.7**. □

Definition 2.10.9. We call a three-manifold obtained in this way a **Heegaard splitting**.

Proposition 2.10.10. *Every orientable three-manifold has a Heegaard splitting.*

Now we move to homology with coefficients. The general idea is that there is nothing sacred about choosing finite formal sums over **Z**, and in fact, choosing different coefficients may give us more information. We proceed.

Definition 2.10.11. Let G be an abelian group. Then we define $C_n(X;G)$ to be chain group of finite formal sums $\sum g_i\sigma_i$ where $g_i \in G$ and $\sigma_i : \Delta^n \to X$. We still have boundary maps $\partial_n : C_n(X;G) \to C_{n-1}(X;G)$ where $\partial_{n-1}\partial_n = 0$ and we can define relative chains $C_n(X, A; G) = {C_n(X;G)}\big/{C_n(A;G)}$.

[18] This can also be shown using **Poincaré duality 2.13.13**.

Thus we can define **homology with coefficients in** G to be $H_n(X;G) = H_n(C_\bullet(X;G))$ and relative homology with coefficients $H_n(X,A;G) = H_n(C_\bullet(X,A;G))$.

We can also define cellular homology with coefficients just like before. Use the chain complex $\{H_n(X^n, X^{n-1}; G)\}$ and the cellular boundary map is exactly the same:

$$d_n\left(\sum_\alpha g_\alpha {e^n}_\alpha\right) = \sum_{\alpha,\beta} d_{\alpha\beta} g_\alpha {e^{n-1}}_\beta.$$

Then

$$H_n^{CW}(X;G) = {}^{\ker d_n}\!\big/_{\operatorname{Im} d_{n+1}}.$$

We remark that all theorems hold for these homologies: long exact sequences, excision, Mayer-Vietoris, etc.

Example 2.10.12. If X is a point, then

$$H_n(X;G) = \begin{cases} G & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Example 2.10.13. Computed via the long exact sequence and excision, just like before, we have that

$$\widetilde{H_n}(S^k;G) = \begin{cases} G & \text{if } n = k, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 2.10.14. *If $f : S^k \to S^k$ has $\deg(f) = m$, then the induced map $f_* : \widetilde{H_k}(S^k;G) \to \widetilde{H_k}(S^k;G)$ is $f_*(g) = mg$.*

Proof. A homomorphism $\varphi : G_1 \to G_2$ induces a map on relative homology $\varphi_* : H_n(X,A;G_1) \to H_n(X,A;G_2)$ which is natural (**Definition 2.8.52**) with respect to other maps. So, in particular if one group is $\mathbf{Z}$, we have a commutative diagram:

$$\begin{tikzcd}
\mathbf{Z} \cong \widetilde{H_k}(S^k) \arrow[r, "f_*"] \arrow[d, "\varphi_*"] & \widetilde{H_k}(S^k) \arrow[d, "\varphi_*"] \\
G \cong \widetilde{H_k}(S^k;G) \arrow[r, "f_*"] & \widetilde{H_k}(S^k;G)
\end{tikzcd}$$

Then fix $g \in G$ and define $\varphi : \mathbf{Z} \to G$ by $\varphi(1) = g$. Then $f_*(g) = f_*\varphi_*(1) = \varphi_*(f_*(1)) = \varphi_*(m) = mg$, as desired. $\square$

Example 2.10.15. Let's compute the cellular chain complex of $\mathbf{RP}^n$ over $\mathbf{Z}_2$. Recall that $\mathbf{RP}^n = e^n \cup e^{n-1} \cup ... \cup e^0$, so the cellular chain complex is

$$0 \to \mathbf{Z}_2 \to \mathbf{Z}_2 \to \cdots \to \mathbf{Z}_2 \to \mathbf{Z}_2 \to \mathbf{Z}_2 \to 0.$$

The degree of the cellular boundary maps is

$$\deg(id_{S^{k-1}}) + \deg(-id_{S^{k-1}}) = 1 + (-1)^k$$

which is 0 or 2, both $0 \mod 2$.

Thus the maps are

$$0 \to \mathbf{Z}_2 \xrightarrow{0} \mathbf{Z}_2 \xrightarrow{0} \cdots \to \mathbf{Z}_2 \xrightarrow{2\equiv 0} \mathbf{Z}_2 \xrightarrow{0} \mathbf{Z}_2 \to 0.$$

When all maps are the zero map, the homology is the chain complex:

$$\widetilde{H}_k(\mathbf{RP}^n; \mathbf{Z}_2) = \begin{cases} \mathbf{Z}_2 & \text{if } 0 \le k \le n, \\ 0 & \text{otherwise.} \end{cases}$$

In general, when our space is nonorientable, $\mathbf{Z}_2$ is a good group to use.

Example 2.10.16. The usual cellular chain complex for $\mathbf{RP}^n$ is

$$0 \to \mathbf{Z} \xrightarrow{1+(-1)^n} \mathbf{Z} \to \cdots \xrightarrow{0} \mathbf{Z} \xrightarrow{2} \mathbf{Z} \xrightarrow{0} \mathbf{Z} \to 0.$$

The maps are $1 + (-1)^n = \begin{cases} 0 & \text{if } n = 2k+1, k \in \mathbf{Z}, \\ 2 & \text{if } n = 2k, k \in \mathbf{Z}. \end{cases}$ Therefore,

$$H_k(\mathbf{RP}^n) = \begin{cases} \mathbf{Z} & \text{if } n = 0, \\ \mathbf{Z}_2 & \text{if } 0 \le k < n, k \text{ odd}, \\ \mathbf{Z} & \text{if } k = n, \text{ odd}, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 2.10.17. *A loop $f : I \to X$ (think $f : \Delta^1 \to X$) is a singular simplex, and $\partial f = f(1) - f(0) = 0$. Thus a loop is a 1-cycle.*

Therefore, there is a map taking loops $f : I \to X$ to cycles ($\ker(\partial_1 : C_1(X) \to C_0(X))$) to the first homology $H_1(X)$.

This map is well-defined because it is the abelianization of $\pi_1(X)$.

If X is path connected, then $h : \pi_1(X, x_0) \to H_1(X)$ is surjective, and $\ker h$ is the commutator subgroup, so h induces an isomorphism $\pi_1(X, x_0)_{ab} \xrightarrow{\sim} H_1(X)$.

Proof. We show three properties of the map taking $\{f : I \to X \mid f \text{ is a loop}\} \to \ker \partial_1 \to H_1(X)$. First, we show that if f is constant, then $[f] = 0$.

> Let $\sigma : \Delta^2 \to X$ be the constant map where $\sigma(\Delta^2) = f(I)$. Then $\partial\sigma = \sigma|_{[v_1,v_2]} - \sigma|_{[v_0,v_2]} + \sigma|_{[v_0,v_1]} = f - f + f = f$. Thus f is a boundary, and hence it represents a 0 in homology.

Next, we show that if $f \simeq g$, then $[f] = [g]$. This is actually the proof that the map is well-defined.

> Let $F : I \times I \to X$ be a homotopy from f to g. Then subdivide $I \times I$ into simplices σ_1 and σ_2. See the picture:

$$
\begin{tikzcd}
v_2 \arrow[r, "g"] & v_3 \\
v_0 \arrow[u] \arrow[ur] \arrow[r, "f"'] \arrow[ur, phantom, "\sigma_2", near start, shift left=3] \arrow[ur, phantom, "\sigma_1", near end, shift right=3] & v_1 \arrow[u]
\end{tikzcd}
\xrightarrow{F} X.
$$

Then the boundary $\partial(\sigma_1 - \sigma_2) = f + \sigma_1|_{[v_1,v_3]} - \sigma_1|_{[v_0,v_3]} + \sigma_2|_{[v_0,v_3]} - g + \sigma_2|_{[v_0,v_2]}$. Since $\sigma_1|_{[v_0,v_3]} = \sigma_2|_{[v_0,v_3]}$ and $\sigma_1|_{[v_1,v_3]} = \sigma_2|_{[v_0,v_2]}$ which is constant as f is a loop, $\partial(\sigma_1 - \sigma_2) = f - g$, and therefore $[f - g] = [f] - [g] = 0$.

So we do indeed have a map $\pi_1(X, x_0) \to H_1(X)$.

Finally, we show that $[f \cdot g] = [f + g] = [f] + [g]$, which shows that the map is a homomorphism.

See that by the diagram, $\partial\sigma = f + g - f \cdot g$:

$$
\begin{tikzcd}
& v_1 \arrow[dr, "g"] & \\
v_0 \arrow[ur, "f"] \arrow[rr, "f", near start] \arrow[rr, "g", near end] & & v_2
\end{tikzcd}
\xrightarrow{\sigma} X.
$$

$\square$

2.11 Section 11: Cohomology

Definitions: dual chain complex, cohomology, induced map on cohomology, cup product, cohomology ring, tensor product
Main Idea: Cohomology is dual to homology. Computing it is pretty tricky, so the current recommendation is to rely on things like the Universal Coefficient Theorem and Poincaré duality so that you don't have to work from definitions. In this section, we define cohomology, explore the Universal Coefficient Theorem, and study cup products, cohomology rings, and the Künneth Formula's cross products.

We know that H_n is a covariant functor from the category of topological spaces to the category of graded abelian groups. Explicitly, we have $X \mapsto H_n(X)$ and maps $f : X \to Y$ induce to $f_* : H_n(X) \to H_n(Y)$.

Cohomology is backwards; cohomology is a contravariant functor from the category of topological spaces to the category of graded abelian groups. $X \mapsto H^n(X)$, while $f : X \to Y$ maps to $f^* : H^n(Y) \to H^n(X)$.

Why would we want to use cohomology? One of the first motivations is a product structure on homology. We can certainly consider a map $H_i(X) \times H_j(X) \to H_{i+j}(X \times X)$ given by $(e^i, e^j) \mapsto (e^i \times e^j)$, but how would we then map $H_{i+j}(X \times X) \to H_{i+j}(X)$ to get any useful information? Unfortunately, the only natural map is projection onto one of the factors; this results in the trivial homomorphism, and that's no good. But, the contravariance of cohomology comes to the rescue!

In cohomology, we still have $H^i(X) \times H^j(X) \to H^{i+j}(X \times X)$. Now, the diagonal map $\Delta : X \to X \times X$ where $x \mapsto (x, x)$ provides $\Delta^* : H^{i+j}(X \times X) \to H^{i+j}(X)$, resulting in what we'll call the cup product, which turns cohomology into a ring: $H^i(X) \times H^j(X) \xrightarrow{\cup} H^{i+j}(X)$.

So that's where we're heading.

Example 2.11.1. Suppose X is a graph; i.e., in particular, think of X as a 1-dimensional CW or Δ complex. Let G be an abelian group.

Then the cochains are $\Delta^0(X; G)$ and $\Delta^1(X; G)$, where $\Delta^0(X; G)$ is the set of functions f from the vertices of the graph into G; i.e., $\Delta^0(X; G) = \prod\limits_{\text{vertices}} G$, and $\Delta^1(X; G)$ is the set of functions f from the edges to G; i.e., $\Delta^1(X; G) = \prod\limits_{\text{edges}} G$. Note in particular that we use product as opposed to direct sum in homology.

Then there is a natural coboundary map, $\delta : \Delta^0(X; G) \to \Delta^1(X; G)$ defined by if $\varphi \in \Delta^0(X; G)$, then $\delta\varphi([v_0, v_1]) = \varphi(v_1) - \varphi(v_0)$ for an edge $[v_0, v_1]$.

We're going to actually work a few examples before we define cohomology. Bear with me.

Example 2.11.2. Let's say we have a one dimensional complex X; for example:

$$\begin{tikzcd}
& v_1 \arrow[dl] & \\
v_2 & & v_0 \arrow[ul] \arrow[ll]
\end{tikzcd}$$

Let's say that for the purposes of example, $\varphi(v_0) = 5$, $\varphi(v_1) = -2$, and $\varphi(v_2) = 3$. Then the coboundary map $\delta\varphi([v_0, v_1]) = -7$, $\delta\varphi([v_0, v_2]) = -2$, and $\delta\varphi([v_1, v_2]) = 5$.

The cochain complex is

$$0 \leftarrow \Delta^1(X; G) \xleftarrow{\delta} \Delta^0(X; G) \leftarrow 0.$$

Then $H^0(X; G) = \ker \delta$ and $H^1(X; G) = {}^{\Delta^1(X;G)}\!/_{\operatorname{Im}\delta}$. We refer to elements in $\ker \delta$ as cocycles and $\operatorname{Im} \delta$ as coboundaries.

Now, $\varphi \in \ker \delta$ if and only if $\varphi(v_1) = \varphi(v_0)$ for all edges $[v_0, v_1]$, which is the case if and only if φ is constant on all path components. Therefore, $H^0(X; G)$ is the set of functions from path components of X to G; i.e., $H^0(X; G) = \prod\limits_{\text{path components}} G$.

Next, we have $H^1(X; G) = {}^{\Delta^1(X;G)}\!/_{\operatorname{Im}\delta} = 0$ if and only if for all $\psi \in \Delta^1(X; G)$, there exists a solution to $\psi = \delta\varphi$. A solution is unique up to an element of $\ker \delta$; i.e., a constant function. Note that this looks a whole lot like discrete derivatives!

So if X is a tree, then $\psi = \delta\varphi$ always has a solution. Simply pick a vertex x; assign it e.g. 0. Then, since X is a tree, start somewhere and construct φ such that $\psi = \delta\varphi$. There won't be any conflicts because it's a tree and you'll never contradict a previous assignment. Therefore, $H^1(\text{tree}; G) = 0$.

Otherwise, fix a maximal subtree $T \subseteq X$. Then $H^1(X; G) = \prod_{\text{edges } X \setminus T} G$. Note that this is very similar to $H_1(X; G) = \bigoplus_{\text{edges } X \setminus T} G$.

Example 2.11.3. Now let's consider a two dimensional example. Let X be a two dimensional Δ-complex. Then $\Delta^2(X; G)$ is the set of functions f from 2-simplices to G. If ψ is a 1-cochain, i.e., $\psi \in \Delta^1(X; G)$, then $\delta\psi([v_0, v_1, v_2]) = \psi([v_1, v_2]) - \psi([v_0, v_2]) + \psi([v_0, v_1])$. For instance:

$$\begin{tikzcd} & v_2 & \\ v_0 \arrow[ur] \arrow[rr] \arrow[r, phantom, "\circlearrowleft"] & & v_1 \arrow[ul] \end{tikzcd}$$

where $\varphi(v_0) = 0$, $\varphi(v_1) = 5$, and $\varphi(v_2) = 3$, $\delta\varphi([v_0, v_1]) = \psi([v_0, v_1]) = 5$, $\psi([v_0, v_2]) = 3$, and $\psi([v_1, v_2]) = -2$, and $\delta\psi([v_0, v_1, v_2]) = 5 - 2 - 3 = 0$.

Note that it's not surprising that $\delta\psi = 0$, since we did a coboundary twice! Indeed, for $\varphi \in \Delta^0(X; G)$,

$$\begin{aligned} \delta\delta\varphi([v_0, v_1, v_2]) &= (\varphi(v_2 - \varphi(v_1)) - (\varphi(v_2) - \varphi(v_0)) + (\varphi(v_1) - \varphi(v_0)) \\ &= 0. \end{aligned}$$

Thus the cochain complex looks as follows:

$$0 \leftarrow \Delta^2(X; G) \xleftarrow{\delta} \Delta^1(X; G) \xleftarrow{\delta} \Delta^0(X; G) \leftarrow 0.$$

Returning to the ODE analogy, given $\psi \in \Delta^1(X; G)$, we can view $\delta\psi \in \Delta^2(X; G)$ as an obstruction to solving $\psi = \delta\varphi$, since if $\psi = \delta\varphi_0$, then $\delta\psi = \delta\delta\varphi_0 = 0$. If $\delta\psi = 0$, then $\psi = \delta\varphi$ has a solution if and only if $[\psi] \in H^1(X; G)$ is 0.

This is akin to $\overline{F} = \nabla F$ if and only if $\nabla \times \overline{F} = 0$.

Now let's get explicit.

Definition 2.11.4. For a chain complex

$$\cdots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots$$

we can form the **dual chain complex**:

$$\cdots \leftarrow {C_{n+1}}^* \xleftarrow{\delta_n} {C_n}^* \xleftarrow{\delta_{n-1}} {C_{n-1}}^* \leftarrow \cdots$$

where ${C_n}^* = \hom(C_n, G)$ and $\delta_n(f) = f\partial_{n+1}$, i.e., $\delta_n(C_n \xrightarrow{f} G) = C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{f} G$.

Note that $\delta_{n+1}\delta_n f = \delta_{n+1}(f\partial_{n+1}) = f\partial_{n+1}\partial_{n+2} = 0$, so this really is indeed a chain complex. Thus, we can define a homology.

Definition 2.11.5. The **cohomology** of a chain complex $C_\bullet$ is

$$H^n(C_\bullet; G) = H_n(C_\bullet{}^*) = {}^{\ker \delta_n}\!\big/_{\operatorname{Im}\delta_{n-1}},$$

i.e., cocycles mod coboundaries.

Note that in general, $H^n(C_\bullet; G) \neq \hom(H_n(C_\bullet), G)$ even though $C_n{}^* = \hom(C_n; G)$. We will see, however, in the **Universal Coefficient Theorem 2.11.9**, that $H^n(C_\bullet; G)$ can be computed from $H_n(C_\bullet)$ and G.

Example 2.11.6. Consider the cellular chain complex for $\mathbf{RP}^3$:

$$\begin{array}{ccccccccccc} C_\bullet : & 0 & \longrightarrow & \mathbf{Z} & \xrightarrow{0=\partial_3} & \mathbf{Z} & \xrightarrow{0=\partial_2} & \mathbf{Z} & \xrightarrow{0=\partial_1} & \mathbf{Z} & \longrightarrow 0 \\ & & & \parallel & & \parallel & & \parallel & & \parallel & \\ & & & C_3 & & C_2 & & C_1 & & C_0 & \end{array}$$

Then applying $\hom(-, \mathbf{Z})$, we get

$$\begin{array}{ccccccccccc} C_\bullet{}^* : & 0 & \longleftarrow & \mathbf{Z} & \xleftarrow[\delta_2]{} & \mathbf{Z} & \xleftarrow[\delta_1]{} & \mathbf{Z} & \xleftarrow[\delta_0]{} & \mathbf{Z} & \longleftarrow 0 \\ & & & \parallel & & \parallel & & \parallel & & \parallel & \\ & & & C_3{}^* & & C_2{}^* & & C_1{}^* & & C_0{}^* & \end{array}$$

We claim that δ_2 is the zero map. Indeed, given $f \in C_2{}^*$, then for $x \in C_3$, $\delta_2 f(x) = f(\partial_3 x) = f(0) = 0$. Similarly, we can show that δ_0 is the zero map.

However, δ_1 is not. Given $f \in c_1{}^*$, then $\delta_1 f(x) = f(\partial_2 x) = f(2x) = 2f(x)$. Therefore, $\delta_1 = 2$.

Now observe the following:

n	0	1	2	3
$H_n(C_\bullet)$	$\mathbf{Z}$	$\mathbf{Z}_2$	0	$\mathbf{Z}$
$H^n(C_\bullet)$	$\mathbf{Z}$	0	$\mathbf{Z}_2$	$\mathbf{Z}$

We'll see more of this in **Poincaré duality 2.13.13**. For now, notice that in particular, $H^2(C_\bullet) = \mathbf{Z}_2 \neq \hom(H_2(C_\bullet, \mathbf{Z}) = \hom(0, \mathbf{Z}) = 0$.

However, there does exist a map $h : H^n(C_\bullet; G) \to \hom(H_n(C_\bullet), G)$.

Consider $H_n(C_\bullet) = {}^{\ker \partial_n}\!\big/_{\operatorname{Im}\partial_{n+1}} = {}^{Z_n}\!\big/_{B_n}$. A class in $H^n(C_\bullet; G)$ can be represented by a map $\varphi : C_n \to G$ such that $\delta_n \varphi = 0$, i.e., $\varphi\partial_{n+1} = 0$. Hence $\varphi(B_n) = 0$.

We can restrict this map to just the cycles Z_n. Let $\varphi_0 = {}^{\varphi}\!\big/_{Z_n}$. This induces a homomorphism $\overline{\varphi_0} : {}^{Z_n}\!\big/_{B_n} \to G$. If $\varphi \in \operatorname{Im}\delta_{n-1}$, then $\varphi = \delta_{n-1}\psi = \psi\partial_n$, and hence $\varphi(Z_n) = 0$, and so $\overline{\varphi_0} = 0$. Vanishing on coboundaries implies that we have a well-defined homomorphism $h : H^n(C_\bullet; G) \to \hom(H_n(C_\bullet), G)$.

Lemma 2.11.7. *This homomorphism h is surjective.*

Proof. Consider the short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & Z_n \arrow[r, "\iota"] & C_n \arrow[r, "\partial_n"] \arrow[l, dotted, bend left, "p"] & B_{n-1} \arrow[r] \arrow[l, dotted, bend left, "s"] & 0 \end{tikzcd}$$

Since B_{n-1} is a free abelian group, the **Splitting Lemma 2.10.4** gives us maps p and s, and the sequence splits.

In particular, there exists $p : C_n \to Z_n$ such that $p\iota = id_{Z_n}$. So we can extend a homomorphism $\varphi_0 : Z_n \to G$ to a homomorphism $\varphi = \varphi_0 p : C_n \to G$. Therefore, we can extend a homomorphism $H_n(C_\bullet) \to G$ to a homomorphism $C_n \to G$ that vanishes on B_n. Note that this extension gives us a map in the opposite direction; i.e., we have a map $h' : \hom(H_n(C_\bullet), G) \to \ker \delta_n \to H^n(C_\bullet; G)$. One can check that $hh' = id$, so h is surjective (and h' is injective). $\square$

We have shown that h is surjective, but moreover, from the proof we have a split short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \ker h \arrow[r] & H^n(C_\bullet; G) \arrow[r, "h"] & \hom(H_n(C_\bullet), G) \arrow[r] \arrow[l, dotted, bend left, "h'"] & 0 \end{tikzcd}$$

Our goal now is to analyze $\ker h$. Consider the following commutative diagram of sequences:

$$\begin{tikzcd} 0 \arrow[r] & Z_{n+1} \arrow[r, "\iota"] \arrow[d, "0"] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "\partial_{n+1}"] & B_n \arrow[r] \arrow[d, "0"] & 0 \\ 0 \arrow[r] & Z_n \arrow[r, "\iota"] & C_n \arrow[r, "\partial_n"] & B_{n-1} \arrow[r] & 0 \end{tikzcd}$$

When we dualize, we have

$$\begin{tikzcd} 0 & {Z_{n+1}}^* \arrow[l] & {C_{n+1}}^* \arrow[l] & {B_n}^* \arrow[l] & 0 \arrow[l] \\ 0 & {Z_n}^* \arrow[l] \arrow[u, "0"] & {C_n}^* \arrow[l] \arrow[u, "\delta_n"] & {B_{n-1}}^* \arrow[l] \arrow[u, "0"] & 0 \arrow[l] \end{tikzcd}$$

Note that the dualized sequence is still exact, since $\hom(C_n, G) = \hom(Z_n \oplus B_{n-1}, G) = \hom(Z_n, G) \oplus \hom(B_{n-1}, G)$ and ${C_n}^* = \hom(C_n, G) = \hom(Z_n, G) \oplus \hom(B_{n-1}, G) = {Z_n}^* \oplus {B_{n-1}}^*$.

Now consider the long exact sequence associated to the above dualized short exact sequence:

$$\cdots \leftarrow {B_n}^* \xleftarrow{{\iota_n}^*} {Z_n}^* \leftarrow H^n(C_\bullet; G) \leftarrow {B_{n-1}}^* \leftarrow {Z_{n-1}}^*$$

The boundary map ${Z_n}^* \to {B_n}^*$ is dual to the inclusion map $\iota_n : B_n \to Z_n$. In other words, it is restriction to B_n. Thus, we get the following short exact sequences:

$$0 \leftarrow \ker {\iota_n}^* \leftarrow H^n(C_\bullet; G) \leftarrow {B_{n-1}}^* \big/ \operatorname{Im} {\iota_{n-1}}^* \leftarrow 0.$$

Note that $\ker \iota_n{}^* = \hom(H_n(C_\bullet), G)$, as $\ker \iota_n{}^*$ is the set of homomorphisms $\varphi_0 : Z_n \to G$ that vanish on B_n.

We will show that the above sequence splits.

Now consider ${B_{n-1}{}^*}/{\operatorname{Im} \iota_{n-1}{}^*}$, which we will call $\operatorname{coker} \iota_{n-1}{}^*$. Informally, $\operatorname{coker} \iota_{n-1}{}^*$ is the "target mod image." Indeed, $\operatorname{coker} \iota_{n-1}{}^*$ measures the failure of exactness of the dual to the following short exact sequence:

$$0 \to B_{n-1} \xrightarrow{\iota_{n-1}} Z_{n-1} \to H_{n-1}(C_\bullet) \to 0.$$

In other words, $\operatorname{coker} \iota_{n-1}{}^*$ measures the failure of exactness of the following sequence:

$$0 \leftarrow B_{n-1}{}^* \xleftarrow{\iota_{n-1}{}^*} Z_{n-1}{}^* \leftarrow H_{n-1}(C_\bullet)^* \leftarrow 0,$$

as $B_{n-1}{}^* \xleftarrow{\iota_{n-1}{}^*} Z_{n-1}{}^*$ is the only possible place that exactness can fail. Homomorphisms on B may not extend to Z.

Now, the short exact sequence

$$0 \to B_{n-1} \xrightarrow{\iota_{n-1}} Z_{n-1} \to H_{n-1}(C_\bullet) \to 0$$

is a free resolution of $H_{n-1}(C_\bullet)$, which we call H. A free resolution is an exact sequence $F_\bullet$ of the form

$$\cdots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0,$$

where F_i are free.

The dual is a cochain complex, not necessarily exact:

$$\cdots \leftarrow F_2{}^* \xleftarrow{f_2{}^*} F_1{}^* \xleftarrow{f_1{}^*} F_0{}^* \xleftarrow{f_0{}^*} H^* \leftarrow 0.$$

Then $H^n(F_\bullet; G) = H_n(F_\bullet{}^*) = {\ker f_{n+1}{}^*}/{\operatorname{Im} f_n{}^*}$, and we have $\operatorname{coker} \iota_{n-1}{}^* = {B_{n-1}{}^*}/{\operatorname{Im} \iota_{n-1}{}^*} = H^1(F_\bullet; G)$.

Lemma 2.11.8. *For any two free resolutions $F_\bullet$ and $F_\bullet{}'$ of H, there is a canonical isomorphism $H^n(F_\bullet; G) \xrightarrow{\sim} H^n(F_\bullet{}'; G)$.*

Proof. We sketch the proof. The idea is to construct a commutative diagram by induction:

$$\begin{tikzcd}
\cdots \arrow[r] & F_2 \arrow[r, "f_2"] \arrow[d, "\alpha_2"] & F_1 \arrow[r, "f_1"] \arrow[d, "\alpha_1"] & F_0 \arrow[r, "f_0"] \arrow[d, "\alpha_0"] & H \arrow[r] \arrow[d, equal] & 0 \\
\cdots \arrow[r] & F_2{}' \arrow[r, "f_2{}'"] & F_1{}' \arrow[r, "f_1{}'"] & F_0{}' \arrow[r, "f_0{}'"] & H \arrow[r] & 0
\end{tikzcd}$$

Then define α_0 by fixing a basis $\{x_\beta\}$ for F_0 and for each x_β, choose $x_\beta{}' \in F_0{}'$ such that $f_0{}'(x_\beta{}') = f_0(x_\beta)$. Set $\alpha_0(x_\beta) = x_\beta{}'$. Now the first square commutes, as desired.

Now for α_1, notice that $f_0{}'\alpha_0 f_1 = f_0 f_1 = 0$. Thus $\alpha_0 f_1(x) \in \ker f_0{}' = \operatorname{Im} f_1{}'$. Then again, fix a basis $\{x_\beta\}$ for F_1. For each x_β, choose $x_\beta{}'$ in $F_1{}'$ such that $f_1{}'(x_\beta{}') = \alpha_0 f_1(x_\beta)$. Set $\alpha_1(x_\beta) = x_\beta{}'$. Now the second square commutes, as desired.

Induct. It remains to be seen that $\alpha_n{}^* : H^n(F_\bullet{}'; G) \to H^n(F_\bullet; G)$ is an isomorphism, and does not depend on the choices made. $\square$

Now, for an abelian group H, we always have the free resolution

$$\cdots \to 0 \to F_1 \to F_0 \to H \to 0,$$

where F_0 has basis of that of H and F_1 is the kernel of $F_0 \to H$. Then the only interesting cohomology group of this complex will be $H^1(F_\bullet; G)$, since, as we have said, this is the only place exactness can fail.

Let $H^1(F_\bullet; G) = \operatorname{Ext}(H, G)$. By **Lemma 2.11.8**, $\operatorname{Ext}(H, G)$ only depends on H and G. The name comes as $\operatorname{Ext}(H, G)$ is related to extensions: $0 \to G \to K \to H \to 0$.

Now, we are ready to state the following:

Theorem 2.11.9 (**Universal Coefficient Theorem for Cohomology**). *There exists a split short exact sequence*

$$0 \to \operatorname{Ext}(H_{n-1}(C_\bullet), G) = \operatorname{coker} \iota_{n-1}{}^* \to H^n(C_\bullet; G) \to \hom(H_n(C_\bullet), G) \to 0.$$

In other words, $H^n(C_\bullet; G) = \hom(H_n(C_\bullet), G) \oplus \operatorname{Ext}(H_{n-1}(C_\bullet), G)$.

We now list some properties which make computing $\operatorname{Ext}(H, G)$ easier:

1. $\operatorname{Ext}(H \oplus H', G) = \operatorname{Ext}(H, G) \oplus \operatorname{Ext}(H', G)$, since the sum of free resolutions is free.

2. $\operatorname{Ext}(H, G) = 0$ if H is free, since we can use the resolution $0 \to H \to H \to 0$.

3. $\operatorname{Ext}(\mathbf{Z}_n, G) \cong G/nG$, since we can use the resolution $0 \to \mathbf{Z} \xrightarrow{n} \mathbf{Z} \to \mathbf{Z}_n \to 0$, which dualizes to $0 \leftarrow G \xleftarrow{n} G \leftarrow \mathbf{Z}_n{}^* \leftarrow 0$.

Corollary 2.11.10. *If $H_n(C_\bullet)$ and $H_{n-1}(C_\bullet)$ are finitely generated with torsion subgroups $T_n \subseteq H_n(C_\bullet)$ and $T_{n-1} \subseteq H_{n-1}(C_\bullet)$, then*

$$H^n(C_\bullet; \boldsymbol{Z}) = \hom(H_n(C_\bullet), \boldsymbol{Z}) \oplus \operatorname{Ext}(H_{n-1}(C_\bullet), \boldsymbol{Z}) = H_n(C_\bullet)/T_n \oplus T_{n-1}.$$

Note that sequences are natural (**Definition 2.8.52**) with respect to the chain map $C_\bullet \to C_\bullet{}'$. Thus, we have another corollary to the **Universal Coefficient Theorem 2.11.9**:

Corollary 2.11.11. *If $\alpha : C_\bullet \to C_\bullet{}'$ is a chain map that induces an isomorphism on homology $\alpha_* : H_n(C_\bullet) \to H_n(C_\bullet{}')$, then the induced map on cohomology $\alpha^* : H^n(C_\bullet{}'; G) \to H^n(C_\bullet; G)$ is also an isomorphism.*

We also make the following remarks:

- $H^0(X; G) = \hom(H_0(X), G)$. This can actually be shown via definitions, but the **Universal Coefficient Theorem 2.11.9** easily implies it, as $\operatorname{Ext}(H_{-1}(X), G) = 0$.
- $H^1(X; G) = \hom(H_1(X), G)$. This can too be shown directly, but the **Universal Coefficient Theorem 2.11.9** implies it, as $\operatorname{Ext}(H_0(X), G) = 0$, since $H_0(X)$ is free.
- If X is path connected, then $\hom(H_1(X), G) = \hom(\pi_1(X), G)$.

Definition 2.11.12. Given a map $f : X \to Y$, f **induces** a map $f^* : H^n(Y; G) \to H^n(X; G)$ by $f^\# \varphi = \varphi f$ for $\varphi \in C^n(Y; G)$.

Proposition 2.11.13. *The induced map f^* is well-defined, and only depends on the homotopy class of f.*

The proof involves the prism operator.

• • •

We now turn to explicitly computing the cohomology groups of spaces, now that we have some good machinery under our belts. We also remark that all homology theorems apply, with reversed arrows, to cohomology. In other words, we get reduced cohomology, relative cohomology, the long exact sequence

$$\begin{tikzcd}
 & \cdots & H^{n+1}(X,A;G) \arrow[l] \\
H^n(A;G) \arrow[urr] & H^n(X;G) \arrow[l] & H^n(X,A;G) \arrow[l] \\
H^{n-1}(A;G) \arrow[urr] & \cdots \arrow[l] ,
\end{tikzcd}$$

homotopy invariance, excision, Mayer-Vietoris, and so on.

Example 2.11.14. Let X be the Klein bottle. We compute the cohomology with $\mathbf{Z}$ and $\mathbf{Z}_m$ coefficients. The Klein bottle with Δ-complex structure is pictured below.

$$\begin{tikzcd}[column sep=huge, row sep=huge]
v \arrow[r, "b"] & v \arrow[d, "a"] \\
v \arrow[u, "a"] \arrow[r, "b"'] \arrow[ur, "c"] \arrow[ur, phantom, "U"', shift right=6] \arrow[ur, phantom, "L", shift left=6] & v
\end{tikzcd}$$

We have the cochain complex

$$\begin{array}{ccccccccc}
0 & \longleftarrow & \Delta^2(X;G) & \xleftarrow[\delta_1]{} & \Delta^1(X;G) & \xleftarrow[\delta_0]{} & \Delta^0(X;G) & \longleftarrow & 0 \\
 & & \| & & \| & & \| & & \\
 & & G^2 & & G^3 & & G & & \\
 & & \| & & \| & & \| & & \\
 & & \langle U^*, L^* \rangle & & \langle a^*, b^*, c^* \rangle & & \langle v^* \rangle & &
\end{array}$$

Now, we compute $\delta_0 : \Delta^0(X;G) \to \Delta^1(X;G)$. See that

$$\begin{aligned}
\delta_0 v^*(n_1 a + n_2 b + n_3 c) &= v^*(\partial_1(n_1 a + n_2 b + n_3 c)) \\
&= n_1 v^* \partial_1(a) + n_2 v^* \partial_1(b) + n_3 v^* \partial_1(c) \\
&= 0,
\end{aligned}$$

since $\partial_1 = 0$. Thus $\delta_0 v^* = 0$, so $\delta_0 = 0$.

Check out when we use matrix form: $\delta_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} : G \to G^3$. Note that $\partial_1 : \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} : G^3 \to G$, so ${\partial_1}^T = \delta_0$.

Next, we compute $\delta_1 : \Delta^1(X;G) \to \Delta^2(X;G)$, by computing it on the basis elements. First,

$$\begin{aligned}
\delta_1 a^*(n_1 U + n_2 L) &= n_1 a^* \partial_2(U) + n_2 a^* \partial_2(L) \\
&= n_1 a^*(a + b - c) + n_2 a^*(a - b + c) \\
&= n_1 + n_2.
\end{aligned}$$

Thus, $\partial_1 a^* = U^* + L^*$. Next,

$$\begin{aligned}
\delta_1 b^*(n_1 U + n_2 L) &= n_1 b^* \partial_2(U) + n_2 b^* \partial_2(L) \\
&= n_1 b^*(a + b - c) + n_2 b^*(a - b + c) \\
&= n_1 - n_2,
\end{aligned}$$

so $\delta_1 b^* = U^* - L^*$. Finally,

$$\begin{aligned}
\delta_1 c^*(n_1 U + n_2 L) &= n_1 c^* \partial_2(U) + n_2 c^* \partial_2(L) \\
&= n_1 c^*(a + b - c) + n_2 c^*(a - b + c) \\
&= -n_1 + n_2,
\end{aligned}$$

and $\delta_1 c^* = -U^* + L^*$.

If we use matrix form, $\delta_1 = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} : G^3 \to G^2$. Again, note that $\partial_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 1 \end{bmatrix} : G^2 \to G^3$, and ${\partial_2}^T = \delta_1$!

Now, we have our maps δ_0 and δ_1, so the cohomology is

$$H^0(X;G) = {}^{\ker \delta_0}\!/_{\operatorname{Im} \delta_{-1}} = {}^{G}\!/_{0} = G.$$

For $H^1(X;G)$, see that $\begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 0 \end{bmatrix}$. Then

$$\ker \delta_1 = \begin{cases} \langle b^* - c^* \rangle & \text{if } G = \mathbf{Z} \text{ or 2 does not divide } m, \\ \left\langle b^* - c^*, \frac{m}{2}(a^* - b^*) \right\rangle & \text{if 2 divides } m. \end{cases}$$

Note that to compute this, you're looking for things to make both rows 0.

Therefore,

$$\begin{aligned} H^1(X;G) &= {}^{\ker \delta_1}\!/_{\operatorname{Im} \delta_0} \\ &= {}^{\ker \delta_1}\!/_{0} \\ &= \begin{cases} G & \text{if } G = \mathbf{Z} \text{ or 2 does not divide } m, \\ G \times \mathbf{Z}_2 & \text{if 2 divides } m, \end{cases} \end{aligned}$$

since $\langle a^* - b^*, b^* - c^*, c^* \rangle$ is a basis for G^3.

Finally, for $H^2(X;G)$, see that $\operatorname{Im} \delta_1 = \langle U^* + L^*, U^* - L^* \rangle = \langle 2U^*, U^* - L^* \rangle \subseteq \langle U^*, U^* - L^* \rangle = G^2$. Therefore,

$$\begin{aligned} H^2(X;G) &= {}^{G^2}\!/_{\operatorname{Im} \delta_1} \\ &= {}^{\langle U^*, U^* - L^* \rangle}\!/_{\langle 2U^*, U^* - L^* \rangle} \\ &= \begin{cases} 0 & \text{if 2 does not divide } m, \\ \mathbf{Z}_2 & \text{if } G = \mathbf{Z} \text{ or 2 divides } m. \end{cases} \end{aligned}$$

Therefore,

$$H^k(X;\mathbf{Z}) = \begin{cases} \mathbf{Z} & \text{if } k = 0, 1, \\ \mathbf{Z}_2 & \text{if } k = 2, \\ 0 & \text{otherwise,} \end{cases}$$

$$H^k(X;\mathbf{Z}_{2n}) = \begin{cases} \mathbf{Z}_{2n} & \text{if } k = 0, \\ \mathbf{Z}_{2n} \times \mathbf{Z}_2 & \text{if } k = 1, \\ \mathbf{Z}_2 & \text{if } k = 2, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$H^k(X;\mathbf{Z}_{2n+1}) = \begin{cases} \mathbf{Z}_{2n+1} & \text{if } k = 0, 1, \\ 0 & \text{otherwise.} \end{cases}$$

Now we turn to the cup product, a way to turn cohomology into a ring.

Definition 2.11.15. Let R be a ring. The **cup product** is

$$\cup : C^k(X;R) \times C^l(X;R) \to C^{k+l}(X;R),$$

defined by:

Given some singular $(k+l)$-simplex $\sigma : \Delta^{k+l} \to X$, if $\varphi \in C^k(X;R)$, $\psi \in C^l(X;R)$, then set

$$(\varphi \cup \psi)(\sigma) = \varphi\left(\sigma|_{[v_0,\ldots,v_k]}\right)\psi\left(\sigma|_{[v_k,\ldots,v_{k+l}]}\right)$$

with the multiplication in R.

Example 2.11.16. Let $\sigma : [v_0, v_1, v_2] \to X$. See the picture.

$$\begin{tikzcd} & v_2 & & \\ v_0 \arrow[ur] \arrow[rr, "\varphi"'] & & v_1 \arrow[ul, "\psi"'] & \xrightarrow{\sigma} X. \end{tikzcd}$$

Then $\varphi, \psi \in C^1(X;R)$, $\varphi \cup \psi \in C^2(X;R)$, and

$$(\varphi \cup \psi)(\sigma) = \varphi\left(\sigma|_{[v_0,v_1]}\right)\psi\left(\sigma|_{[v_1,v_2]}\right).$$

A good intuition to have is to continue thinking about derivatives and products. In particular, we have the following lemma, showing that the cup product is well-defined on cohomology, which looks like the Leibniz product rule.

Lemma 2.11.17. $\delta(\varphi \cup \psi) = \delta\varphi \cup \psi + (-1)^k \varphi \cup \delta\psi$.

This implies that the induced function

$$\cup^* : H^k(X;R) \times H^l(X;R) \to H^{k+l}(X;R),$$

where $[\varphi] \cup [\psi] \mapsto [\varphi \cup \psi]$*, is well-defined.*

Proof. Let's look at a few cases first:

- If $\delta\varphi = \delta\psi = 0$, then $\delta(\varphi \cup \psi) = 0$; i.e., the cup of cocycles is a cocycle.
- If $\delta\varphi = 0$, i.e., φ is a cocycle, then $\pm\varphi \cup \delta\psi = \delta(\varphi \cup \psi)$. In other words, cocycle cup coboundary is a coboundary.
- If $\delta\psi = 0$, then $\delta\varphi \cup \psi = \delta(\varphi \cup \psi)$. So coboundary cup cocycle is coboundary.

Thus, succinctly put, the product of cocycle and coboundary is a coboundary. Now let's prove the lemma. Let $\sigma : \Delta^{k+l+1} \to X$. Then

$$\begin{aligned} \delta\varphi \cup \psi(\sigma) &= \sum_{i=0}^{k+1} (-1)^i \varphi\left(\sigma|_{[v_0,\ldots,\hat{v}_i,\ldots,v_{k+1}]}\right) \cdot \psi\left(\sigma|_{[v_{k+1},\ldots,v_{k+l+1}]}\right) \\ &= \delta\varphi\left(\sigma|_{[v_0,\ldots,v_{k+1}]}\right) \cdot \psi\left(\sigma|_{[v_{k+1},\ldots,v_{k+l+1}]}\right) \end{aligned}$$

and

$$
\begin{aligned}
(-1)^k \varphi \cup \delta\psi(\sigma) &= \sum_{i=k}^{k+l+1} (-1)^i \varphi\left(\sigma|_{[v_0,\ldots,v_k]}\right) \cdot \psi\left(\sigma|_{[v_k,\ldots,\hat{v_i},\ldots,v_{k+l+1}]}\right) \\
&= \sum_{i=k}^{k+l+1} (-1)^i \varphi\left(\sigma|_{[v_0,\ldots,v_k]}\right) \cdot \delta\psi\left(\sigma|_{[v_k,\ldots,v_{k+l+1}]}\right).
\end{aligned}
$$

If we add $\left(\delta\varphi \cup \psi + (-1)^k \varphi \cup \delta\psi\right)(\sigma)$, the last term in $\delta\varphi \cup \psi(\sigma)$ is

$$
(-1)^{k+1} \varphi\left(\sigma|_{[v_0,\ldots,v_k]}\right) \psi\left(\sigma|_{[v_{k+1},\ldots,v_{k+l+1}]}\right)
$$

and the first term in $(-1)^k \varphi \cup \delta\psi(\sigma)$ is

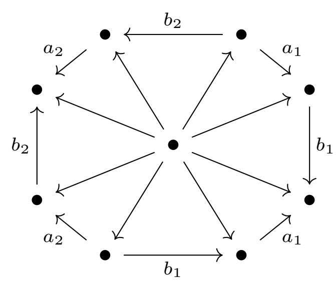
$$
(-1)^k \varphi\left(\sigma|_{[v_0,\ldots,v_k]}\right) \psi\left(\sigma|_{[v_{k+1},\ldots,v_{k+l+1}]}\right).
$$

Thus they cancel, and

$$
\begin{aligned}
\left(\delta\varphi \cup \psi + (-1)^k \varphi \cup \delta\psi\right)(\sigma) &= \sum_{i=0}^{k+l+1} (-1)^i (\varphi \cup \psi)\left(\sigma|_{[v_0,\ldots,\hat{v_i},\ldots,v_{k+l+1}]}\right) \\
&= \delta(\varphi \cup \psi),
\end{aligned}
$$

as claimed. Therefore the cup product is well-defined on cohomology. $\square$

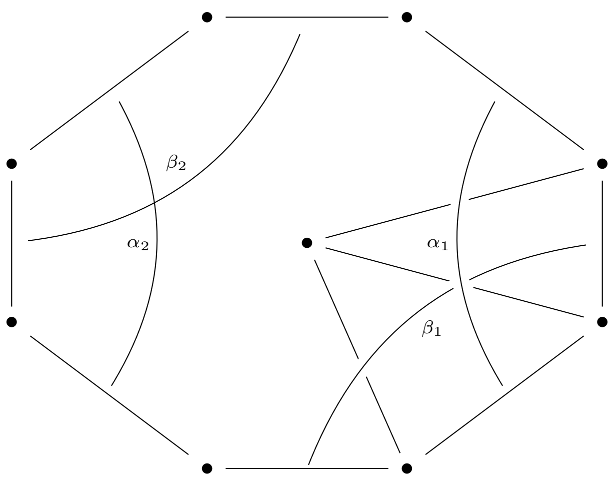
Example 2.11.18. Let $X = S_g$. Then triangulate X to get a Δ-complex structure. For example, if $g = 2$:



The a_is and b_is give a basis for $H_1(X) = \mathbf{Z}^{2g}$. Since $H_0(X)$ is always free, $H^1(X;\mathbf{Z}) = \hom(H_1(X), \mathbf{Z})$. Let's use α_i and β_i for the dual basis:

- $\alpha_i(a_i) = 1$, and $\alpha_i(\text{any other basis element}) = 0$, and
- $\beta_i(b_i) = 1$, and $\beta_i(\text{any other basis element}) = 0$.

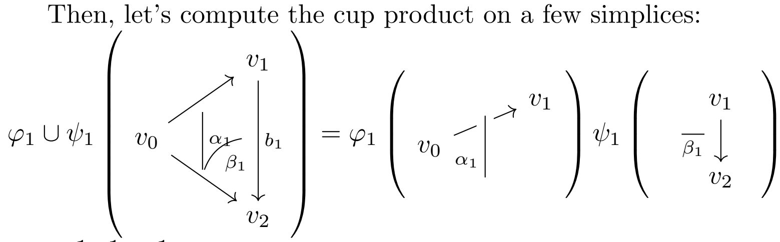
With the same Δ-complex structure, we have



Now the maps φ_i and ψ_i count the intersection with the curves α_i and β_i, respectively, on a given 1-simplex. Note that these are cocycles, since they take value 0 on the boundary of any 2-simplex.

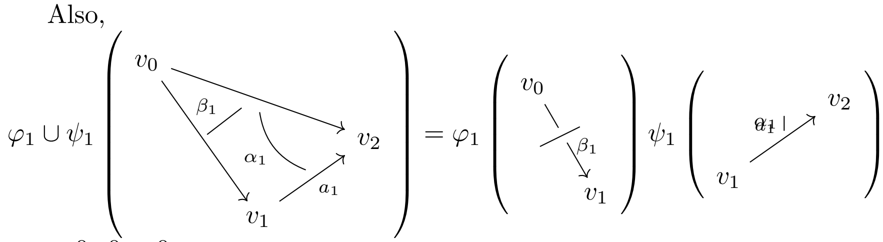
Furthermore, $[\varphi_i] = \alpha_i$ and $[\psi_i] = \beta_i$.

Then, let's compute the cup product on a few simplices:



$= 1 \cdot 1 = 1.$

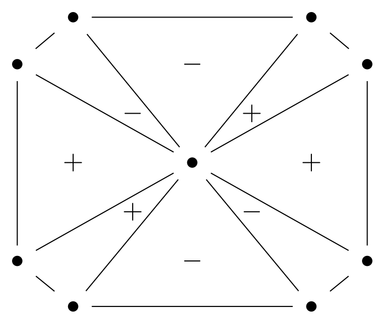
Also,



$= 0 \cdot 0 = 0.$

And $\varphi_1 \cup \psi_1$(any other 2-simplex) $= 0$, since either α_1 or β_1 won't intersect the simplex.

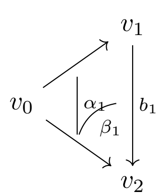
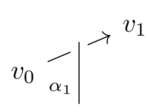
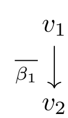
Next, let c be the sum of the 2-simplices with signs as in the following figure.



Then $\partial c = 0$, and $[c] \neq 0 \in H_2(X)$, since $C_3(X) = 0$.

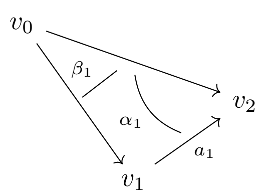
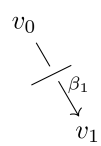
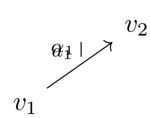
Since $\varphi_1 \cup \psi_1(c) = 1$, we get that $[c]$ is a generator for $H_2(X) = \mathbf{Z}$, and $\gamma = [\varphi_1 \cup \psi_1]$ is a generator for $H^2(X;\mathbf{Z}) = \hom(H_2(X), \mathbf{Z}) = \mathbf{Z}$.

Now we compute

$\psi_1 \cup \varphi_1$ () $= \psi_1$ () φ_1 ()

$= 0 \cdot 0 = 0,$

and the only other interesting cup, as above, is

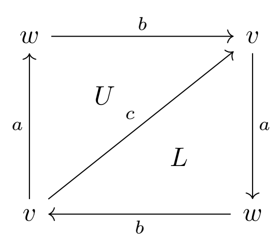
$\psi_1 \cup \varphi_1$ () $= \psi_1$ () φ_1 ()

$= 1 \cdot 1 = 1.$

Thus $\psi_1 \cup \varphi_1(c) = -1$, and therefore $[\psi_1 \cup \varphi_1] = -\gamma$.

In general, we find that, being sloppy with notation, $\alpha_i \cup \beta_j$ is γ if $i = j$ and 0 otherwise. We also get anticommutivity: $\alpha_i \cup \beta_j = -\beta_j \cup \alpha_i$. Also, $\alpha_i \cup \alpha_j = \beta_i \cup \beta_j = 0$.

Example 2.11.19. Let's compute cup products on $\mathbf{RP}^2$. We have the Δ-complex



Then we know

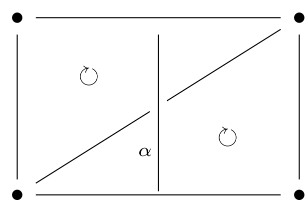
$$H^k(\mathbf{RP}^2; \mathbf{Z}) = \begin{cases} \mathbf{Z} & \text{if } k = 0, \\ \mathbf{Z}_2 & \text{if } k = 2, \\ 0 & \text{otherwise.} \end{cases}$$

The cup product structure here is not interesting, since adding any dimensions of cells hits a 0 group. Thus, we're going to look at the cup product stucture on $H^k(\mathbf{RP}^2; \mathbf{Z}_2)$. The complex is $0 \to {\mathbf{Z}_2}^2 \to {\mathbf{Z}_2}^3 \to {\mathbf{Z}_2}^2 \to 0$, and

$$H^k(\mathbf{RP}^2; \mathbf{Z}_2) = \begin{cases} \mathbf{Z}_2 & \text{if } k = 0, 1, 2, \\ 0 & \text{otherwise.} \end{cases}$$

We'll get something nontrivial here.

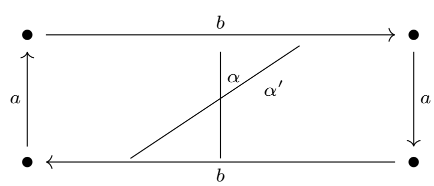
Let $\alpha = [a+b] \in H_1(\mathbf{RP}^2; \mathbf{Z}_2)$. Let α^* be the dual to $[a+b]$ in $H^1(\mathbf{RP}^2; \mathbf{Z}_2) = \hom(H_1(\mathbf{RP}^2; \mathbf{Z}_2), \mathbf{Z}_2)$.



Let φ count the mod 2 intersection with α. Then φ is a cocycle mod 2. φ on U is 0, and φ on L is 2. Also, $[\varphi] = \alpha^*$. Then $\varphi \cup \varphi(U) = 0 \cdot 1 = 0$ and $\varphi \cup \varphi(L) = 1 \cdot 1 = 1$.

Notice that $\partial(U + L) = 0$, i.e., $U + L$ is a boundary, and $\varphi \cup \varphi(U + L) = 1$, so $[U+L]$ is a generator for $H_2(\mathbf{RP}^2; \mathbf{Z}_2)$ and $[\varphi \cup \varphi]$ is a generator for $H^2(\mathbf{RP}^2; \mathbf{Z}_2)$.

Note the following: α', homotopic to α and transverse, must intersect α:



Whereas on S_g, it is certainly possible to have α' homotopic to α and disjoint; we can not have intersection.

Proposition 2.11.20. *If $f : X \to Y$, the induced map $f^* : H^n(Y; R) \to H^n(X; R)$ satisfies that f^* is a ring homomorphism; i.e.,*

$$f^*(\alpha \cup \beta) = f^*(\alpha) \cup f^*(\beta).$$

Proof idea. Check on cochains that the formula holds;

$$f^\#(\varphi \cup \psi) = f^\#(\varphi) \cup f^\#(\psi).$$

□

Example 2.11.21. A homeomorphism $f : S_g \to S_g$ induces an automorphism of $H^1(S_g; \mathbf{Z}) = \mathbf{Z}^{2g}$, i.e., a matrix in $GL_{2g}(\mathbf{Z})$.

Does every matrix arise? The answer is no, since a matrix must preserve the cup product structure.

Choose a basis $a_1, b_1, a_2, b_2, ..., a_g, b_g$. Then

$$J = \begin{bmatrix} 0 & 1 & & & & ... & 0 \\ -1 & 0 & & & & ... & 0 \\ & & 0 & 1 & & ... & 0 \\ & & -1 & 0 & & ... & 0 \\ & & & & ... & & \\ 0 & ... & & & & 0 & 1 \\ 0 & ... & & & & -1 & 0 \end{bmatrix}$$

Then the cup product $x \cup y = x^T J y$. For example, if $g = 2$, then $a_1 \cup b_1 = 1$. To see this, note that

$$J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix},$$

so

$$a_1 J b_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = 1$$

In order for a matrix for f^*, A to arise, the matrix A must satisfy $x^T J y = x \cup y = Ax \cup Ay = x^T A^T J A y$. So $J = A^T J A$.

The group of such matrices is called the symplectic group $\mathrm{Sp}(2g, \mathbf{Z})$, and all of these matrices arise.

Theorem 2.11.22. *If R is commutative, then $\alpha \cup \beta = (-1)^{kl} \beta \cup \alpha$ for all $\alpha \in H^k(X; R)$ and $\beta \in H^l(X; R)$.*

Proof. Given $\sigma : [v_0, ..., v_n] \to X$, define $\overline{\sigma} : [v_0, ..., v_n] \xrightarrow{*} [v_n, ..., v_0] \xrightarrow{\sigma} X$.

The map $*$ is $n + (n-1) + ... + 1 = \frac{n(n+1)}{2}$ transpositions. Let $\varepsilon_n = (-1)^{\frac{n(n+1)}{2}}$.

Next, define a homomorphism on the level of chains $\rho : C_n(X) \to C_n(X)$ by $\rho(\sigma) = \varepsilon_n(\overline{\sigma})$. Then we claim that ρ is chain homotopic to the identity; i.e., we claim that there exists $P : C_n(X) \to C_{n+1}(X)$ such that $\partial P + P \partial = \rho - id$. We take this claim on faith, though it is shown in the text. It's essentially the prism operator with order reversed.

Given this claim, for $\varphi \in C^k(X; R)$ and $\psi \in C^l(X; R)$,

$$\begin{aligned} \rho^* \varphi \cup \rho^* \psi(\sigma) &= \rho^* \varphi \left(\sigma|_{[v_0, ..., v_k]} \right) \rho^* \psi \left(\sigma|_{[v_k, ..., v_n]} \right) \\ &= \varphi \left(\varepsilon_k \sigma|_{[v_k, ..., v_0]} \right) \psi \left(\varepsilon_l \sigma|_{[v_n, ..., v_k]} \right), \end{aligned}$$

while

$$\begin{aligned}\rho^*(\psi \cup \varphi)(\sigma) &= \psi \cup \varphi(\rho\sigma) \\ &= \varepsilon_{k+l}\psi\left(\sigma|_{[v_n,\ldots,v_k]}\right)\varphi\left(\sigma|_{[v_k,\ldots,v_0]}\right).\end{aligned}$$

Note that $\varepsilon_{k+l} = (-1)^{kl}\varepsilon_k\varepsilon_l$.

As ρ^* is the identity on cohomology, $[\psi \cup \varphi] = (-1)^{kl}[\varphi] \cup [\psi]$. $\square$

Definition 2.11.23. We define the **cohomology ring** to be

$$H^*(X; R) = \bigoplus_{n \geq 0} H^n(X; R).$$

This is a graded ring; n provides the grading. In fact, the cohomology ring is an R-algebra (i.e., we have scalars in R and can add and multiply).

Example 2.11.24. $H^*(\mathbf{RP}^n; \mathbf{Z}_2) = {}^{\mathbf{Z}_2[\alpha]}\!/_{\alpha^{n+1}}$. The ring is graded by the degree $|\alpha| = 1$; i.e., $\alpha \in H^1(\mathbf{RP}^n; \mathbf{Z}_2)$. We'll prove this in **Theorem 2.11.34**

Example 2.11.25. $H^*(\mathbf{RP}^\infty; \mathbf{Z}_2) = \mathbf{Z}_2[\alpha]$, $|\alpha| = 1$.

Example 2.11.26. $H^*(\mathbf{CP}^n; \mathbf{Z}) = {}^{\mathbf{Z}[\alpha]}\!/_{\alpha^{n+1}}$. $|\alpha| = 2$; i.e., $\alpha \in H^2(\mathbf{CP}^n; \mathbf{Z})$.

Example 2.11.27. $H^*(\mathbf{CP}^\infty; \mathbf{Z}) = \mathbf{Z}[\alpha]$, $|\alpha| = 2$.

We now begin to introduce concepts which we will use to define the **Künneth Formula 2.11.30**, which relates $H^*(X; R) \times H^*(Y; R)$ to $H^*(X \times Y; R)$.

Given a space $X \times Y$, there are natural projections p_X and p_Y. Consider the cross product $H^*(X) \times H^*(Y) \xrightarrow{\times} H^*(X \times Y)$, given by $a \times b = {p_X}^*(a) \cup {p_Y}^*(b)$, where the cup is taken in $H^*(X \times Y)$. The cross product is a bilinear map, and this is never a homomorphism, which we would want.

The solution is to use tensor products instead.

Definition 2.11.28. Suppose A and B are abelian groups. Then the **tensor product** $A \otimes B$ is the abelian group generated by ${}^{a \otimes b}\!/_{\sim}$, where $(a + a') \otimes b \sim a \otimes b + a' \otimes b$ and $a \otimes (b + b') \sim a \otimes b + a \otimes b'$. The zero element is $0 \otimes 0 = 0 \otimes b = a \otimes 0$.[19] Inverses are $-(a \otimes b) = (-a) \otimes b = a \otimes (-b)$.[20]

Example 2.11.29. $\mathbf{Z} \otimes \mathbf{Z} \cong \mathbf{Z}$.

The map is $(a \otimes b) \mapsto ab$. To see that this map is an isomorphism, it is first a homomorphism since the bilinear map $\mathbf{Z} \times \mathbf{Z} \to \mathbf{Z}$, $(a, b) \mapsto ab$, induces the homomorphism given. It is injective, since $\ker((a \otimes b) \mapsto ab) = 0$, and it is surjective, since $(n \otimes 1) \mapsto n$.

We have the following properties of tensor products:

[19] as $a \otimes 0 = a \otimes (0 + 0) = a \otimes 0 + a \otimes 0$.

[20] as $a \otimes b + (-a) \otimes b = (a - a) \otimes b = 0 \otimes b = 0$.

1. $A \otimes B \cong B \otimes A$.
2. $\left(\bigoplus_i A_i\right) \otimes B = \bigoplus_i (A_i \otimes B)$.
3. $(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$.
4. $\mathbf{Z} \otimes A \cong A$ via the map $n \otimes a \mapsto na$ with $a \mapsto 1 \otimes a$ inverse. (See $1 \otimes 2a = 1 \otimes a + 1 \otimes a = 2 \otimes a$.)
5. $\mathbf{Z}_n \otimes A \cong A/nA$.
6. Homomorphisms $f : A \to A'$ and $g : B \to B'$ induce a homomorphism $f \otimes g : A \otimes B \to A' \otimes B'$ by $(f \otimes g)(a \otimes b) = f(a) \otimes g(b)$.
7. A bilinear map $\varphi : A \times B \to C$ induces a homomorphism $\varphi : A \otimes B \to C$ by $\varphi(a \otimes b) = \varphi(a, b)$. Note that this is exactly what we used in the above **Example 2.11.29**.

We also remark the following:

Given R-modules A and B where R is a commutative ring, we define $A \otimes_R B = {A \otimes B}/{(ra) \otimes b = a \otimes (rb)}$. Then $A \otimes_R B$ is an R-module; $r(a \otimes b) = (ra) \otimes b = a \otimes (rb)$.

If A and B are graded R-algebras, then we can define an algebra structure on the product $(a \otimes b)(c \otimes d) = (-1)^{|b|\cdot|c|} ac \otimes bd$, with $|b|$ and $|c|$ the degrees of b and c. And $A \otimes_R B$ is also a graded R-algebra. Thus, we can state the following:

Theorem 2.11.30 (**Künneth Formula**). *If X and Y are CW-complexes and $H^k(Y; R)$ is a finitely generated R-module for all k, then the cross product $H^*(X; R) \otimes_R H^*(Y; R) \to H^*(X \times Y; R)$ is an isomorphism of graded R-algebras.*

Proof. We sketch the proof. Fix the space Y and define the functors $h^n(X) = \bigoplus_{i+j=n} H^i(X; R) \otimes_R H^j(Y; R)$ and $k^n(X) = H^n(X \times Y; R)$. We wish to show that these two functors are equal. To do this, one must show

1. h^* and k^* satisfy the axioms of cohomology theory,
2. the cross product defines a natural map $\mu : h^n(X) \to k^n(X)$, and
3. if $\mu : h^*(\text{point}) \to k^*(\text{point})$ is an isomorphism, then $\mu : h^*(X) \to k^*(X)$ is an isomorphism for all CW-complexes X. $\square$

Example 2.11.31. Recall **Example 2.11.24**. See that

$$\begin{aligned} H^*(\mathbf{RP}^n \times \mathbf{RP}^n; \mathbf{Z}_2) &\cong H^*(\mathbf{RP}^n; \mathbf{Z}_2) \otimes_{\mathbf{Z}_2} H^*(\mathbf{RP}^n; \mathbf{Z}_2) \\ &= \mathbf{Z}_2[\alpha_1] / {\alpha_1}^{n+1} \otimes_{\mathbf{Z}_2} \mathbf{Z}_2[\alpha_2] / {\alpha_2}^{n+1} \\ &= \mathbf{Z}_2[\alpha_1, \alpha_2] / ({\alpha_1}^{n+1}, {\alpha_2}^{n+1}). \end{aligned}$$

Example 2.11.32. Since $S^1 = \mathbf{RP}^1$, $H^*(S^1; \mathbf{Z}) = \mathbf{Z}[\alpha] / \alpha^2$ with $|\alpha| = 1$.

Example 2.11.33. Let T^n be the n-torus; i.e., $T^n = \prod_{i=1}^{n} S^1$. Then

$$\begin{aligned} H^*(T^n; \mathbf{Z}) &= \mathbf{Z}[\alpha_1] / {\alpha_1}^2 \otimes_{\mathbf{Z}} \mathbf{Z}[\alpha_2] / {\alpha_2}^2 \otimes_{\mathbf{Z}} \dots \otimes_{\mathbf{Z}} \mathbf{Z}[\alpha_n] / {\alpha_n}^2 \\ &= \Lambda_{\mathbf{Z}}[\alpha_1, \dots, \alpha_n], \end{aligned}$$

which is called the exterior algebra. It is a free module over $\mathbf{Z}$ with basis $\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_k}$, where $i_1 < i_2 < \dots < i_k$. The algebra relations are that ${\alpha_i}^2 = 0$ and $\alpha_i \alpha_j = -\alpha_j \alpha_i$ (which actually implies the first).

For example, if $n = 2$, then $(\alpha_1 \otimes 1)(1 \otimes \alpha_2) = (-1)^{|1| \cdot |1|} \alpha_1 \otimes \alpha_2 = \alpha_1 \otimes \alpha_2$, since $|1| \cdot |1| = 0$. And, $(1 \otimes \alpha_2)(\alpha_1 \otimes 1) = (-1)^{|\alpha_1| \cdot |\alpha_2|} \alpha_1 \otimes \alpha_2 = -\alpha_1 \otimes \alpha_2$, since $|\alpha_1| \cdot |\alpha_2| = 1$.

In particular, $\operatorname{rank} H^i(T^n; \mathbf{Z}) = \binom{n}{i}$, as the degree of each $\mathbf{Z}[\alpha_i] / {\alpha_i}^2$ is either 0 or 1, so we simply choose i of them to be nonzero.

We remark now that we can also define relative cup and cross products. We have

$$\cup : H^k(X, A; R) \times H^l(X, B; R) \to H^{k+l}(X, A \cup B; R)$$

defined by

$$\cup : C^k(X, A; R) \times C^l(X, B; R) \to C^{k+l}(X, A + B; R)$$

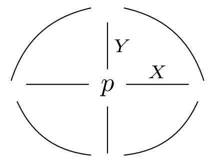
where $C^k(X, A; R)$ are k-cochains on X that vanish on A and $C^{k+l}(X, A+B; R)$ are $(k+l)$-cochains that vanish on sums $\sum \sigma + \sum \sigma'$ for $\sum \sigma \subseteq A$ and $\sum \sigma' \subseteq B$. As before, $C^n(X, A \cup B; R) \hookrightarrow C^n(X, A + B; R)$ gives an isomorphism on H^*.

Similarly, $\times : H^k(X, A; R) \times H^l(Y, B; R) \to H^{k+l}(X \times Y, A \times Y \cup X \times B; R)$.

Theorem 2.11.34. *$H^*(\boldsymbol{RP}^n; \boldsymbol{Z}_2) \cong \boldsymbol{Z}_2[\alpha] / \alpha^{n+1}$.*

Proof. The inclusion $\mathbf{RP}^{n-1} \hookrightarrow \mathbf{RP}^n$ induces an isomorphism on the cohomology H^i for $i \leq n-1$, so it suffices to show that the cup product of a generator for $H^1(\mathbf{RP}^n; \mathbf{Z}_2)$ and a generator for $H^{n-1}(\mathbf{RP}^n; \mathbf{Z}_2)$ is a generator for $H^n(\mathbf{RP}^n; \mathbf{Z}_2)$.

Let $X = \{\pm(x_0, x_1, 0, \dots, 0) \mid {x_0}^2 + {x_1}^2 = 1\} = \mathbf{RP}^1 = S^1$. Let $Y = \{\pm(0, x_1, \dots, x_n) \mid {x_1}^2 + \dots + {x_n}^2 = 1\} = \mathbf{RP}^{n-1}$. Then $X \cap Y = \{\pm(0, 1, 0, \dots, 0)\}$; call this point p. Let $U = \{\pm(x_0, x_1, \dots, x_n) \mid {x_0}^2 + \dots + {x_n}^2 = 1, x_1 \neq 0\}$, which is homeomorphic to $\mathbf{R}^n$. In fact, $U = \mathbf{R}^1 \times \mathbf{R}^{n-1} = (x_0) \times (x_2, \dots, x_n)$, and $\overline{0}$ corresponds to the point p. Consider the picture when $n = 2$; we have:



We have U the interior of the disk.

Note that in general, $\mathbf{RP}^n = U \cup \mathbf{RP}^{n-1}$; this is the standard cell structure. Now we have the diagram

$$\begin{tikzcd}
H^1(\mathbf{RP}^n;\mathbf{Z}_2) \times H^{n-1}(\mathbf{RP}^n;\mathbf{Z}_2) \arrow[r, "\cup"] & H^n(\mathbf{RP}^n;\mathbf{Z}_2) \\
H^1(\mathbf{R}^n, \mathbf{R}^n \setminus \mathbf{R}^{n-1};\mathbf{Z}_2) \times H^{n-1}(\mathbf{R}^n, \mathbf{R}^n \setminus \mathbf{R}^1;\mathbf{Z}_2) \arrow[u] \arrow[r, "\cup"] & H^n(\mathbf{R}^n, \mathbf{R}^n \setminus \{0\};\mathbf{Z}_2) \arrow[u]
\end{tikzcd}$$

We claim that the maps are isomorphism and the diagram commutes by naturality of the cup product. To see this, the map on the right

$$H^n(\mathbf{R}^n, \mathbf{R}^n \setminus \{0\};\mathbf{Z}_2) \xrightarrow{\sim} H^n(\mathbf{RP}^n, \mathbf{RP}^n \setminus \{p\};\mathbf{Z}_2)$$

is an isomorphism by excision; we remove $\mathbf{RP}^{n-1}$. By deformation retraction, we have from the long exact sequence an isomorphism

$$H^n(\mathbf{RP}^n, \mathbf{RP}^n \setminus \{p\};\mathbf{Z}_2) \xrightarrow{\sim} H^n(\mathbf{RP}^n, \mathbf{RP}^{n-1};\mathbf{Z}_2),$$

and from cellular cohomology, an isomorphism

$$H^n(\mathbf{RP}^n, \mathbf{RP}^{n-1};\mathbf{Z}_2) \xrightarrow{\sim} H^n(\mathbf{RP}^n;\mathbf{Z}_2),$$

as desired.

The one on the left is similar.

As for the bottom row, it is just the cross product in disguise:

$$H^1(\mathbf{R}, \mathbf{R} \setminus \{0\};\mathbf{Z}_2) \times H^{n-1}(\mathbf{R}^{n-1}, \mathbf{R}^{n-1} \setminus \{0\};\mathbf{Z}_2) \to H^n(\mathbf{R}^n, \mathbf{R}^n \setminus \{0\};\mathbf{Z}_2),$$

since $\mathbf{R}^n = \mathbf{R}^1 \times \mathbf{R}^{n-1}$ has projections p_1 and p_2 such that ${p_1}^*$ and ${p_2}^*$ induce isomorphisms on H^*. Apply relative Künneth, and you're done. □

2.12 Section 12: Orientations

Definitions: orientation (on $\mathbf{R}^n$), local orientation, local homology, orientation (on a manifold), R-orientation, fundamental class

Main Idea: Orientablility is a well-defined choice of generator of $H_n(M \mid x; R)$. Every manifold is $\mathbf{Z}_2$-orientable.

Definition 2.12.1. An **orientation on $\mathbf{R}^n$ at a point** x is a choice of generator for $H_n(\mathbf{R}^n, \mathbf{R}^n \setminus \{x\}) \cong \mathbf{Z}$.

This isomorphism comes from the long exact sequence for the pair $(\mathbf{R}^n, \mathbf{R}^n \setminus \{x\})$:

$$\widetilde{H_n}(\mathbf{R}^n, \mathbf{R}^n \setminus \{x\}) \cong H_{n-1}(\mathbf{R}^n \setminus \{x\}) \cong H_{n-1}(S^{n-1}) \cong \mathbf{Z}.$$

This determines an orientation at any $y \in \mathbf{R}^n$ by

$$H_n(\mathbf{R}^n, \mathbf{R}^n \setminus \{x\}) \xleftarrow{\sim} H_n(\mathbf{R}^n, \mathbf{R}^n \setminus B) \xrightarrow{\sim} H_n(\mathbf{R}^n, \mathbf{R}^n \setminus \{y\}).$$

This comes from the long exact sequence for the triple $(\mathbf{R}^n, \mathbf{R}^n \setminus \{x\}, \mathbf{R}^n \setminus B)$, where B is an open ball containing x and y. Since $\mathbf{R}^n \setminus \{x\}$ and $\mathbf{R}^n \setminus B$ are deformation retracts, their relative homology vanishes.

Definition 2.12.2. Let M be an n-manifold. Fix $x \in M$, and fix an open neighborhood $U \subseteq M$ of x with a homeomorphism $\varphi : U \to \mathbf{R}^n$. Then, via excision, we have a homeomorphism

$$H_n(M, M \setminus \{x\}) \xrightarrow{\sim} H_n(U, U \setminus \{x\}),$$

and via φ_*, we have

$$H_n(U, U \setminus \{x\}) \xrightarrow{\sim} H_n(\mathbf{R}^n, \mathbf{R}^n \setminus \{\varphi(x)\}) \cong \mathbf{Z}.$$

We define: a **local orientation** of M at x is a choice of generator for $H_n(M, M \setminus \{x\}) \cong \mathbf{Z}$.

Definition 2.12.3. We define **local homology of X at A** with the following notation:

$$H_n(X, X \setminus A) = H_n(X \mid A)$$
$$H_n(X, X \setminus A; G) = H_n(X \mid A; G).$$

Definition 2.12.4. An **orientation of an n-manifold M at a point x** is a function $x \mapsto \mu_x \in H_n(M \mid x)$, with μ_x a generator, that is "locally consistent."

A function is locally consistent if for all $x \in M$, there is a neighborhood $U \cong_\varphi \mathbf{R}^n$ of x and an open ball B such that $x \in B \subseteq U$ and for all $y \in B$,

$$H_n(M \mid x) \xleftarrow{\sim} H_n(M \mid B) \xrightarrow{\sim} H_n(M \mid y);$$

i.e., thereis a generator $\mu_B \in H_n(M \mid B)$ such that $\mu_B \mapsto \mu_y$ for all $y \in B$.

We say that a manifold is orientable if it has an orientation.

Theorem 2.12.5. *Every manifold has a* 2*-sheeted cover that is orientable.*

Proof. Let $\widetilde{M} = \{\mu_x \mid x \in M, \mu_x \text{ is a local orientation.}\}$. Then the map $\mu_x \mapsto x$ is two-to-one. The topology on $\widetilde{M}$ is as follows:

Given an open ball $B \subseteq U \cong \mathbf{R}^n$ and a generator $\mu_B \in H_n(M \mid B)$, let $U(\mu_B)$ be the set of μ_x for $x \in B$ that are the image of μ_B under $H_n(M \mid B) \xrightarrow{\sim} H_n(M \mid x)$. This is a basis (**Definition 1.13.1**) for a topology on $\widetilde{M}$, as one can check.

Now we show that $\widetilde{M}$ is orientable. See that $\mu_x \mapsto \widetilde{\mu_x} \in H_n(\widetilde{M} \mid \mu_x)$. Via excision, $H_n(\widetilde{M} \mid \mu_x) \cong H_n(U(\mu_B) \mid \mu_x)$. Then $H_n(U(\mu_B) \mid \mu_x) \cong H_n(B \mid x) \cong H_n(M \mid x)$, and $\mu_x \in H_n(M \mid x)$. ☐

Example 2.12.6. Two examples of orientable 2-sheeted covers are $p : S^2 \to \mathbf{RP}^2$ and p : torus $\to$ Klein bottle. We'll see that both $\mathbf{RP}^2$ and the Klein bottle are not ($\mathbf{Z}$-)orientable.

Corollary 2.12.7. *If M is connected, then M is orientable if and only if $\widetilde{M}$, as described in the proof of **Theorem 2.12.5**, has two components (informally, one oriented positively, one oriented negatively).*

We can actually generalize orientation (which is actually $\mathbf{Z}$-orientation) to any ring R. Let's see how to do that.

Given a manifold M, we can define $M_{\mathbf{Z}} = \{\alpha_x \mid x \in M, \alpha_x \in H_n(M \mid x)\}$. Now we get an infinite sheeted cover $\alpha_x \mapsto x$. Notice that restricting $\alpha_x = 0$, we get a subcover $M_0 \cong M$, and restricting $\alpha_x = \pm k$, we get a subcover $M_k \cong \widetilde{M}$.

Definition 2.12.8. A **section** is a map $M \to M_{\mathbf{Z}}$ such that $x \mapsto \alpha_x$.

An orientation on M is the same thing as a section $x \mapsto \alpha_x \in M_1 \cong \widetilde{M}$. In other words, a section is a continuous choice of local orientation.

Thus, to generalize, let R be a commutative ring with identity[21]. Then the long exact sequence gives us that

$$H_n(M \mid x; R) \cong H_{n-1}(S^{n-1}; R) \cong R,$$

and we define $M_R = \{\alpha_x \mid x \in M, \alpha_x \in H_n(M \mid x; R)$. We get a covering space $M_R \to M$. Then

Definition 2.12.9. An **R-orientation** is a section $M \to M_R$, $x \mapsto \alpha_x \in H_n(M \mid x; R)$ where α_x is a generator such that $\alpha_x R = R$; i.e., α_x is a unit.

We remark that using the isomorphism

$$H_n(M \mid x; R) \cong R \cong \mathbf{Z} \otimes R \cong H_n(M \mid x) \otimes R,$$

we have that each $r \in R$ determines a subcover M_r consisting of the points $\pm\mu_x \otimes r$ for generators $\pm\mu_x \in H_n(M \mid x)$.

If r has order 2, then $r = -r$, so $M_r \cong M$. Otherwise, $M_r = M_{-r} \cong \widetilde{M}$.

Note the following:

"Orientable" as defined in **Definition 2.12.4** is exactly "$\mathbf{Z}$-orientable." Furthermore, if a manifold is $\mathbf{Z}$-orientable, then it is R-orientable for all R, since M_u has a component homeomorphic to M, there exists a section $M \to M_u$. Finally, note that every manifold is $\mathbf{Z}_2$-orientable, as $M_{\mathbf{Z}_2} = M_0 \cup M_1 \cong M \cup M$.

Proposition 2.12.10. *Let M be a closed (i.e., compact with no boundary) connected n-manifold. Then*

1. *if M is R-orientable, then $H_n(M; R) \to H_n(M \mid x; R) \cong R$ is an isomorphism.*

[21] For our purposes, this will almost always be $\mathbf{Z}$ or $\mathbf{Z}_2$.

2. *if M is not R-orientable, then $H_n(M;R) \to H_n(M \mid x;R) \cong R$ is injective, with image $\{r \mid 2r = 0\}$.*

Note what this says about the top-level nth homology group:

$$H_n(M) = \begin{cases} \mathbf{Z} & \text{if } M \text{ is orientable,} \\ 0 & \text{if } M \text{ is not orientable,} \end{cases}$$

and $H_n(M;\mathbf{Z}_2) = \mathbf{Z}_2$.

Definition 2.12.11. A **fundamental class** is an element $\mu \in H_n(M;R)$ whose image in $H_n(M \mid x;R)$ is a unit for all $x \in M$.

By **Proposition 2.12.10**, if M is R-orientable, then there exists a fundamental class. Conversely, if there exists a fundamental class $\mu \in H_n(M;R)$, then the map $\mu \mapsto \mu_x$ is a section of $M_R \to M$, so M is R-orientable.

2.13 Section 13: Cohomology II: Cap Product, Poincaré Duality

Definitions: cap product, (simplicial) cohomology with compact supports, cohomology with compact supports, direct limit of groups, relative cap product
Main Idea: Poincaré duality is a powerful way of comparing the homology and cohomology groups of an n-dimensional R-orientable closed manifold. If M is such a manifold, then $H^k(M;R) \cong H_{n-k}(M;R)$.

Definition 2.13.1. We define the **cap product**. For $k \geq l$,

$$\cap : C_k(X;R) \times C^l(X;R) \to C_{k-l}(X;R)$$

given by, if $\sigma : \Delta^k \to X \in C_k(X;R)$ and $\varphi \in C^l(X;R)$, then

$$\sigma \cap \varphi = \varphi\left(\sigma|_{[v_0,\dots,v_l]}\right) \sigma|_{[v_l,\dots,v_k]} \in C_{k-l}(X;R).$$

We call $\varphi\left(\sigma|_{[v_0,\dots,v_l]}\right)$ the coefficient and $\sigma|_{[v_l,\dots,v_k]}$ is a $k-l$ simplex; $\Delta^{k-l} \to X$.

Lemma 2.13.2. *The cap product is well-defined on homology; that is,*

$$\partial(\sigma \cap \varphi) = (-1)^l(\partial\sigma \cap \varphi - \sigma \cap \delta\varphi).$$

Proof. As usual, it's just computation. The right hand side gives us

$$\begin{aligned}\partial\sigma \cap \varphi = &\sum_{i=0}^{l} (-1)^i \varphi\left(\sigma|_{[v_0,\dots,\hat{v_i},\dots,v_{l+1}]}\right) \sigma|_{[v_{l+1},\dots,v_k]} \\ &+ \sum_{i=l+1}^{k} (-1)^i \varphi\left(\sigma|_{[v_0,\dots,v_l]}\right) \sigma|_{[v_l,\dots,\hat{v_i},\dots,v_k]},\end{aligned}$$

and

$$\sigma \cap \delta\varphi = \sum_{i=0}^{l+1} (-1)^i \varphi\left(\sigma|_{[v_0,\dots,\hat{v_i},\dots,v_{l+1}]}\right) \sigma|_{[v_{l+1},\dots,v_k]}.$$

Therefore,

$$\begin{aligned}
\partial\sigma \cap \varphi - \sigma \cap \delta\varphi &= (-1)^l \varphi\left(\sigma|_{[v_0,\dots,v_l]}\right) \sigma|_{[v_{l+1},\dots,v_k]} \\
&\qquad + \sum_{i=l+1}^{k} (-1)^l \varphi\left(\sigma|_{[v_0,\dots,v_l]}\right) \sigma|_{[v_l,\dots,\hat{v_i},\dots,v_k]} \\
&= \sum_{i=l}^{k} (-1)^l \varphi\left(\sigma|_{[v_0,\dots,v_l]}\right) \sigma|_{[v_l,\dots,\hat{v_i},\dots,v_k]} \\
&= (-1)^l \partial(\sigma \cap \varphi),
\end{aligned}$$

as desired.

Thus if $\partial\sigma = 0$ (i.e., σ is a cycle) and $\delta\varphi = 0$ (i.e., φ is a cocycle), then $\partial(\sigma \cap \varphi) = 0$ ($\sigma \cap \varphi$ is a cycle). If $\partial\sigma = 0$, then $\partial(\sigma \cap \varphi) = \pm\sigma \cap \delta\varphi$. And if $\delta\varphi = 0$, then $\partial(\sigma \cap \varphi) = \pm\partial\sigma \cap \varphi$. Together, this tells us that $\cap$ is well-defined on $H_k(X;R) \times H^l(X;R) \to H_{k-l}(X;R)$. $\square$

We remark that we have naturality (**Definition 2.8.52**) in the cap product. If $f : X \to Y$, then we have induced maps so that the diagram commutes:

$$\begin{array}{ccccc}
H_k(X;R) & \times & H^l(X;R) & \xrightarrow{\cap} & H_{k-l}(X;R) \\
\downarrow{\scriptstyle f_*} & & {\scriptstyle f^*}\uparrow & & \downarrow{\scriptstyle f_*} \\
H_k(Y;R) & \times & H^l(Y;R) & \xrightarrow{\cap} & H_{k-l}(Y;R)
\end{array}$$

and $f_*(\alpha) \cap \varphi = f_*(\alpha \cap f^*(\varphi)) \in H_{k-l}(Y;R)$ for $\alpha \in H_k(X;R)$, $\varphi \in H^l(Y;R)$.

• • •

Our goal now is to prove **Poincaré duality 2.13.13**. We'll state it now and show a few examples, build up some machinery, and then state it again and prove it.

Poincaré Duality: If M is a closed, R-orientable n-manifold with fundamental class $[M] \in H_n(M;R)$, then the map $D : H^k(M;R) \to H_{n-k}(M;R)$ defined by $D([\varphi]) = [M] \cap [\varphi]$ is an isomorphism for all k.

Example 2.13.3. Consider the space S_2. In **Example 2.11.18**, we drew and discussed the Δ-complex there, so recall it.

Let c be the 2-chain with assigned signs; then $[c]$ is the fundamental class of the manifold, as it is the generator for $H_2(S_2)$. Then edges a_1, b_1, a_2, b_2 are cycles, and they generate $H_1(S_2)$. Furthermore, $\alpha_1, \beta_1, \alpha_2, \beta_2$ generate $H^1(S_2)$ (with maps $\varphi_1, \psi_1, \varphi_2, \psi_2$ that counter respective intersections).

It can be seen that $[v_0, v_1, v_2] \cap \varphi_1 = \varphi_1([v_0, v_1]) \cdot [v_1, v_2]$, so $c \cap \varphi_1 = b_1$. Likewise, $c \cap \varphi_2 = b_2$, and $c \cap \psi_1 = -a_1$ and $c \cap \psi_2 = -a_2$.

Therefore, in general, $[S_g] \cap \varphi_i = b_i$ and $[S_g] \cap \psi_i = -a_i$.

Notice that this is opposite of the isomorphism from the **Universal Coefficient Theorem 2.11.9**:

$$H^1(S_g; \mathbf{Z}) \cong \hom(H_1(S_g), \mathbf{Z}).$$

Note also that the homotopicity of α_i and b_i are the same, as are a_i and $-\beta_i$.

Definition 2.13.4. Let X be a locally compact simplicial complex; i.e., any simplex is the face of finitely many other simplices. Define $\Delta^i_C(X; G)$ to be the set of cochains $\varphi \in \Delta^i(X; G)$ such that $\varphi(\sigma) \neq 0$ for only finitely many σ. Then observe that $\delta(\Delta^i_C(X; G) \subseteq \Delta^{i+1}_C(X; G)$, as $\delta\varphi(\sigma) \neq 0$ only if φ is nonzero on some face in $\partial\sigma$, by local compactness.

Then we can define **(simplicial) cohomology with compact supports** to be

$$H^i_C(X; G) = {}^{\ker(\delta : \Delta^i_C(X;G) \to \Delta^{i+1}_C(X;G))}\Big/_{\mathrm{Im}(\delta : \Delta^{i-1}_C(X;G) \to \Delta^i_C(X;G))}.$$

Example 2.13.5. Let $X = \mathbf{R}$, with 0-simplices the integer points $\mathbf{Z}$ and 1-simplices the intervals connecting them.

Then, let $\varphi \in \Delta^0_C(X; \mathbf{Z})$. Then $\delta\varphi = 0$ if and only if φ is constant, so φ must be 0. Therefore, $H^0_C(\mathbf{R}; \mathbf{Z}) = 0$.

However, $H^1_C(\mathbf{R}; \mathbf{Z}) \neq 0$. To see this, note that, for instance,

$$\varphi(e) = \begin{cases} 1 & \text{if } e = [0,1] \in \Delta^1_C(\mathbf{R}; \mathbf{Z}), \\ 0 & \text{otherwise.} \end{cases}$$

We claim that $0 \neq [\varphi] \in H^1_C(\mathbf{R}; \mathbf{Z})$. In fact, we claim that $[\varphi]$ generates $H^1_C(\mathbf{R}; \mathbf{Z})$, so $H^1_C(\mathbf{R}; \mathbf{Z}) = \mathbf{Z}$.

To see this, if $\varphi = \delta\psi$, then $\psi(1) - \psi(0) = 1$, and $\psi(n) - \psi(1) = 0$ for $n \geq 1$ and $\psi(m) - \psi(0) = 0$ for $m \leq 0$. Therefore,

$$\psi(n) = \begin{cases} 1 & \text{if } n \geq 1 \\ 0 & \text{if } n \leq 0 \end{cases} + C \notin \Delta^0_C(\mathbf{R}; \mathbf{Z}).$$

The picture is that of integrating a bump function φ. The resulting ψ is not compactly supported!

Notice that we do have a "duality;" i.e., we have complementary homology/cohomology groups: $H^1_C(\mathbf{R}; \mathbf{Z}) = \mathbf{Z} = H_0(\mathbf{R})$, and $H^0_C(\mathbf{R}; \mathbf{Z}) = 0 = H_1(\mathbf{R})$. We'll actually see this in general.

We can define cohomology with compact supports in general, not just for simplices:

Definition 2.13.6. Let $C^i_C(X;G)$ be the set of cochains $\varphi \in C^i(X;G)$ such that there exists a compact set $K_\varphi \subseteq X$ and $\varphi(\sigma) = 0$ if $\sigma(\Delta^i) \subseteq X \setminus K_\varphi$.

Then $\delta(C^i_C(X;G) \subseteq C^{i+1}_C(X;G)$, as $\delta\varphi$ vanishes on $(i+1)$-chains in $X \setminus K_\varphi$.

Then define **cohomology with compact support** as before (cocycles mod coboundaries):

$$H^i_C(X;G) = {\ker(\delta : C^i_C(X;G) \to C^{i+1}_C(X;G))}\big/{\mathrm{Im}(\delta : C^{i-1}_C(X;G) \to C^i_C(X;G))}.$$

Note that if X is compact, then $H^i_C(X;G) = H^i(X;G)$; simply take the compact set to be everything.

We also remark the following alternate description for the compact cochain group:

$$C^i_C(X;G) = \bigcup_{K \text{ compact}} C^i(X, X \setminus K; G).$$

The cochain group $C^i(X, X \setminus K; G)$ is exactly the set of cochains that vanish on $X \setminus K$.

If we have an inclusion map $K \hookrightarrow L$, then we get an induced map $C^i(X, X \setminus K; G) \to C^i(X, X \setminus L; G)$, and so there exists an induced map on the level of cohomology:

$$H^i(X, X \setminus K; G) \to H^i(X, X \setminus L; G).$$

Our goal will be to describe the cohomology with compact supports using this data. To that end:

Definition 2.13.7. Let $\{G_\alpha\}_{\alpha \in I}$ be an indexed collection of groups, where I is a directed set; i.e., for all $\alpha, \beta \in I$, there exists γ such that $\alpha, \beta \leq \gamma$.

We define the property of semiflow, a result of I being a directed set:

- For all $\alpha \leq \beta$, there exists $f_{\alpha\beta} : G_\alpha \to G_\beta$.
- If $\alpha \leq \beta \leq \gamma$, then the following diagram commutes:

$$\begin{tikzcd} G_\alpha \arrow[rr, "f_{\alpha\gamma}"] \arrow[dr, "f_{\alpha\beta}"'] & & G_\gamma \\ & G_\beta \arrow[ur, "f_{\beta\gamma}"'] & \end{tikzcd}$$

  i.e., $f_{\alpha\gamma} = f_{\beta\gamma} f_{\alpha\beta}$.

Then we define the **direct limit of groups** as

$$\lim_{\to} G_\alpha = {\sqcup G_\alpha}\big/{\sim},$$

where $a \sim b$ if $a \in G_\alpha$, $b \in G_\beta$, and there exists γ with $\alpha, \beta \leq \gamma$ such that $f_{\alpha\gamma}(a) = f_{\beta\gamma}(b)$.

The group structure is that if $a \in G_\alpha$ and $b \in G_\beta$, then $[a] + [b] = [a' + b']$, where $a' = f_{\alpha\gamma}(a)$ and $b' = f_{\beta\gamma}(b)$ for $\alpha, \beta \leq \gamma$.

The idea of the relation is that two elements are equal in $\varinjlim G_\alpha$ if they eventually become equal in the directed system.

Lemma 2.13.8. *$H_i(X) = \varinjlim H_i(K)$, where the limit is over compact sets $K \subseteq X$. In other words, the direct limit of groups wouldn't have told us anything new with regards to homology.*

Proof. The inclusion $K \hookrightarrow X$ induces a map $H_i(K) \to H_i(X)$. This map is surjective, as a cycle in $H_i(X)$ can be represented by a finite sum of simplices, hence in some compact set K. This map is also injective, as if a cycle K is a boundary in X, it is the boundary of a chain in a larger compact set L. $\square$

Lemma 2.13.9. *Analogously, $H^i_C(X; G) = \varinjlim H^i(X, X \setminus K; G)$.*

Proof. Every element $\varphi \in \varinjlim H^i(X, X \setminus K; G)$ is represented by a cocycle in $C^i(X, X \setminus K; G)$ for some compact set $K \subseteq X$. Such a cocycle is zero in $\varinjlim H^i(X, X \setminus K; G)$ if and only if it is the coboundary of a cochain in $C^{i-1}(X, X \setminus L; G)$ for some $L \subseteq X$ compact with $K \subseteq L$. Therefore, $H^i_C(X; G) = \varinjlim H^i(X, X \setminus K; G)$, as desired. $\square$

Example 2.13.10. We compute $H^*_C(\mathbf{R}^n; G)$. It suffices to consider compact sets $B(0, r)$ with $r \geq 0$. Any compact set will eventually be contained in one of these balls.

Then via a long exact sequence, $H^i(\mathbf{R}^n, \mathbf{R}^n \setminus B(0, r); G) \cong \widetilde{H^{i-1}}(\mathbf{R}^n \setminus B(0, r); G)$. Via homotopy equivalence,

$$\widetilde{H^{i-1}}(S^{n-1}; G) = \begin{cases} G & \text{if } i = n, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, $H^n(\mathbf{R}^n, \mathbf{R}^n \setminus B(0, r); G) \xrightarrow{\sim} H^n(\mathbf{R}^n, \mathbf{R}^n \setminus B(0, r+1); G)$. Then

$$H^i_C(\mathbf{R}^n; G) = \begin{cases} G & \text{if } i = n, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that this is exactly $H_{n-i}(\mathbf{R}^n)$. Hints of **Poincaré duality 2.13.13** keep appearing. Indeed, we will use direct limits to discuss Poincaré duality.

Definition 2.13.11. We can define the **relative cap product** to be the map

$$H_k(X, A; R) \times H^l(X, A; R) \to H_{k-l}(X; R).$$

The cap product

$$\cap : C_k(X; R) \times C^l(X, A; R) \to C_{k-l}(X; R),$$

where $C^l(X, A; R)$ are the cochains that vanish on A, vanishes on $C_k(A; R)$, so it induces a map

$$C_k(X, A; R) \times C^l(X, A; R) \to C_{k-l}(X; R).$$

Recall from **Lemma 2.13.9** and **Definition 2.12.3** that

$$H^k_C(M;R) = \varinjlim H^k(M, M \setminus K; R) = \varinjlim H^k(M \mid K; R).$$

Then for $K \subseteq L \subseteq M$, naturality of the (relative) cap product gives that

$$\begin{array}{ccccc} H_n(M \mid L;R) & \times & H^k(M \mid L;R) & \xrightarrow{\cap} & H_{n-k}(M;R) \\ \downarrow{\scriptstyle \iota_*} & & {\scriptstyle \iota^*}\uparrow & & \| \\ H_n(M \mid K;R) & \times & H^k(M \mid K;R) & \xrightarrow{\cap} & H_{n-k}(M;R) \end{array}$$

Now let $\mu_K \in H_n(M \mid K;R)$ and $\mu_L \in H_n(M \mid L;R)$ be the images of the fundamental class (**Definition 2.12.11**) for M. Then $\iota_*(\mu_L) = \mu_K$. Naturality gives $\iota_*(\mu_L) \cap x = \mu_L \cap \iota^*(x)$; i.e., $\mu_K \cap x = \mu_L \cap \iota^*(x)$.

Thus the homomorphisms $H^k(M \mid K;R) \to H_{n-k}(M;R)$ given by $x \mapsto \mu_K \cap x$ induce a homomorphism $D_M : H^k_C(M;R) \to H_{n-k}(M;R)$.

Proposition 2.13.12. *If $M = U \cup V$ where U and V are open, then the Mayer-Vietoris sequence gives the following diagram:*

$$\begin{array}{ccccccccccc} \cdots \to & H^k_C(U\cap V) & \longrightarrow & H^k_C(U) \oplus H^k_C(V) & \longrightarrow & H^k_C(M) & \longrightarrow & H^{k-1}_C(U \cap V) & \longrightarrow \cdots \\ & \downarrow{\scriptstyle D_{U\cap V}} & & \downarrow{\scriptstyle D_U \oplus -D_V} & & \downarrow{\scriptstyle D_M} & & \downarrow{\scriptstyle D_{U\cap V}} & \\ \cdots \to & H_{n-k}(U\cap V) & \to & H_{n-k}(U) \oplus H_{n-k}(V) & \to & H_{n-k}(M) & \to & H_{n-k-1}C(U \cap V) & \to \cdots \end{array}$$

which has exact rows and is commutative, up to sign.

Theorem 2.13.13 (**Poincaré Duality**). *The homomorphism*

$$D_M : H^k_C(M;R) \to H_{n-k}(M;R)$$

is an isomorphism.

Proof. We make a few remarks before we begin.

First, if $M = U \cup V$ and $D_U, D_V, D_{U\cap V}$ are all isomorphisms, then by **Proposition 2.13.12** and the **Five Lemma 2.8.53**, so is D_M.

Second, if M is the union of U_is such that $U_1 \subseteq U_2 \subseteq ...$, and each D_{U_i} is an isomorphism, then so is D_M. To see this,

$$H^k_C(U_i) = \varinjlim_{\substack{K \subseteq U_i \\ K \text{ compact}}} H^k(U_i, U_i \setminus K).$$

Then, via excision,

$$\varinjlim_{\substack{K \subseteq U_i \\ K \text{ compact}}} H^k(U_i, U_i \setminus K) = \varinjlim_{\substack{K \subseteq U_i \\ K \text{ compact}}} H^k(M, M \setminus K).$$

Thus making U_i bigger means more sets in the direct limit; i.e., by inclusion there exists a map $H^k_C(U_i) \to H^k_C(U_{i+1})$. Thus $\varinjlim H^k_C(U_i) = H^k_C(M)$, as every compact set K is in some U_i.

Note that here we've proved

$$\varinjlim_{U_i} H_{n-k}(U_i) = H_{n-k}(M);$$

then we have

$$\begin{array}{c} \varinjlim H^k_C(U_i) = H^k_C(M) \\ \Big\downarrow D_{U_i}(\cong) \\ \varinjlim H_{n-k}(U_i) = H_{n-k}(M) \end{array}$$

Then the map D_M induced by the D_{U_i}s is an isomorphism.

Now we outline the proof; we will show

1. Poincaré duality holds when $M = \mathbf{R}^n$,
2. Poincaré duality holds when $M = U \subseteq \mathbf{R}^n$ open, and
3. Poincaré duality holds for a general M.

First, we show that if $M = \mathbf{R}^n$, then $H^k_C(\mathbf{R}^n) \cong H_{n-k}(\mathbf{R}^n)$. We'll explicitly check this.

> Let $B_r = B_d(0, r)$, and let $\sigma_r : \Delta^n \to \mathbf{R}^n$ be a singular n-simplex satisfying:
>
> 1. σ is an orientation-preserving homeomorphism; i.e., $\det([v_1 - v_0 \mid v_2 - v_0 \mid ... \mid v_n - v_0]) > 0$,
> 2. $0 \in \sigma_r(\Delta^n)$, and
> 3. $\partial\sigma_r \subseteq \mathbf{R}^n \setminus B_r$.
>
> Pictorially, e.g. for $n = 2$, σ_r is a triangle completely engulfing $B_d(0, r)$.
>
> Then we have the quotient map $j_r : C_n(\mathbf{R}^n) \to C_n(\mathbf{R}^n, \mathbf{R}^n \setminus B_r)$, where if $r' \geq r$, then $[j_r(\sigma_{r'})] = \mu_r \in H_n(\mathbf{R}^n \mid B_r)$ is a generator. For $r' \geq r$, define $j_{r,r'} : C_n(\mathbf{R}^n, \mathbf{R}^n \setminus B_{r'}) \to C_n(\mathbf{R}^n, \mathbf{R}^n \setminus B_r)$ by $(j_{r,r'})_* \mu_{r'} = \mu_r$, as σ_r and $\sigma_{r'}$ are homologous.
>
> Then define $D_{\mathbf{R}^n,r} : H_n(\mathbf{R}^n \mid B_r) \times H^n(\mathbf{R}^n \mid B_r) \xrightarrow{\cap} H_0(\mathbf{R}^n)$ by $D_{\mathbf{R}^n,r}(x) = \mu_r \cap x$. It is enough to check that this map is an isomorphism. To see this, let $x \in H^n(\mathbf{R}^n \mid B_r)$ and fix $\overline{\varphi} \in C^n(\mathbf{R}^n \mid B_r) = \hom\left({C_n(\mathbf{R}^n)}/{C_n(\mathbf{R}^n \setminus B_r)}, R\right)$ such that $[\overline{\varphi}] = x$. We can then extend it to $\varphi \in C^n(\mathbf{R}^n = \hom(C_n(\mathbf{R}^n), R)$.

Then, $D_{\mathbf{R}^n}(x) = \mu_r \cap x = [\sigma_r \cap \varphi]$. Let $\varphi(\sigma)$ be the signed intersection between $\partial\sigma$ and $\rho = \{(x, 0, ..., 0) \mid x \geq 0\}$. Then $\varphi(\sigma) = 1$. We can restrict to get $\overline{\varphi} \in C^n(\mathbf{R}^n, \mathbf{R}^n \setminus B_r)$.

For $x = [\overline{\varphi}]$, we have $\mu_r \cap x = \left[\varphi(\sigma_r)\, \sigma_r|_{[v_n]}\right] = [\sigma_r(v_n)]$, which is a generator for $H_0(\mathbf{R}^n)$. Since this holds for all r, we have an isomorphism which passes through the direct limit, and thus $D_{\mathbf{R}^n}$ is an isomorphism, as desired.

Now, we need to show that if M is an open set in $\mathbf{R}^n$, then Poincaré duality holds.

Write $M = \bigcup U_i$, where each U_i is a convex open set. This union is actually countable, since $\mathbf{R}^n$ is separable (**Theorem 1.30.7**) and thus second countable (countable basis) (**Definition 1.30.4**).

Now set $V_j = \bigcup\limits_{j<i} U_j$. Recall that we remarked that if $M = U \cup V$ and $D_U, D_V, D_{U\cap V}$ are isomorphisms, then so is D_M. Apply this to $V_{i+1} = V_i \cup U_i$:

Both V_i and $V_i \cap U_i$ are the unions of at most $i-1$ convex sets, so by induction on the number of convex open sets in the cover, Poincaré duality holds for $V_i, U_i, V_i \cap U_i$. Therefore, D_{V_i} is an isomorphism for all i. Now recall that we remarked that if M is the union of $U_1 \subseteq U_2 \subseteq ...$ and D_{U_i} is an isomorphism, then so is D_M. Apply this to D_{V_i} to see that D_M is an isomorphism, as desired.

Finally, we show Poincaré duality for the general case.

If $M = \bigcup U_i$ with each $U_i \cong \mathbf{R}^n$, then we can play the exact same game as the previous step; simply replace convex with open.

If, however, M cannot be written this way, we use **Zorn's Lemma**. There exists a $U \subseteq M$ such that U is the maximal open set such that Poincaré duality holds. Suppose $x \in M \setminus U$. Choose a neighborhood V of x such that $V \cong \mathbf{R}^n$ and $U \cap V = \emptyset$. Then, Poincaré duality holds for U, V, and $U \cap V = \emptyset$. Thus, by our first remark, Poincaré duality holds for $U \cup V$. But this contradicts the maximality of U.

Thus, Poincaré duality has been proven. □

Corollary 2.13.14. *If M is a closed manifold of odd dimension, then $\chi(M) = 0$.*

Proof. If M is orientable, then $\operatorname{rank} H_i(M) = \operatorname{rank} H^{n-i}(M)$ by **Poincaré duality 2.13.13**. Then, the **Universal Coefficient Theorem 2.11.9** says that $\operatorname{rank} H^{n-i}(M) = \operatorname{rank} H_{n-i}(M)$. Therefore $\chi(M) = \sum(-1)^i \operatorname{rank} H_i(M) = 0$, since the terms cancel in pairs, since n is odd.

If, however, M is not orientable, then $2\chi(M) = \chi(\widetilde{M})$ where $\widetilde{M}$ is the orientable double cover (see **Theorem 2.12.5**). By above, $\chi(\widetilde{M}) = 0$. □

Example 2.13.15. Let S be a closed, connected surface. Suppose $S = \partial M$ where M is a compact 3-manifold. Let $N = M \sqcup_S M$. Then N is a closed 3-manifold. Therefore,

$$0 = \chi(N) = \chi(M) + \chi(M) - \chi(S).$$

Thus $\chi(S) = 2\chi(M)$, so $\chi(S)$ is even.

Thus, for instance, there is no 3-manifold for which $\mathbf{RP}^2$ is the boundary.

2.14 Section 14: Higher Homotopy Groups

Definitions: higher homotopy groups, n-connected, fibration
Main Idea: We generalize the fundamental group to map n-spheres into our space. There are a few nice properties of π_n that we explore.

Definition 2.14.1. Let I^n be the n-cube; i.e., $I^n = \{(x_1, ..., x_n) \in \mathbf{R}^n \mid 0 \leq x_i \leq 1\}$. Then $\partial I^n = \{(x_1, ..., x_n) \in I^n \mid \text{ at least one } x_i \in \{0,1\}\}$. We define the **$n$th homotopy group of X based at** x_0, written $\pi_n(X, x_0)$, to be the group of homotopy classes of maps $(I^n, \partial I^n) \to (X, x_0)$. Homotopies must satisfy that $f_t(\partial I^n) = x_0$ for all t; i.e., the boundary stays fixed.

For $n = 1$, we have the usual fundamental group.

The definition also makes sense for $n = 0$, though $\pi_0(X)$ is in general not a group. Rather, we have $I^0 = \{x\}$ and $\partial I^0 = \emptyset$, and two maps $f_0, f_1 : (I^0, \emptyset) \to (X, x_0)$ are homotopic if and only if there is a path between $f_0(x)$ and $f_1(x)$. The path is the homotopy. So $\pi_0(X, x_0)$[22] is the set of path components of X. There's no natural group structure on $\pi_0(X)$, as we have said[23], but this is not the case for $\pi_n(X)$, $n \geq 1$.

For $n \geq 2$, we define a sum on $\pi_n(X, x_0)$:

$$(f+g)(s_1, ..., s_n) = \begin{cases} f(2s_1, s_2, ..., s_n) & \text{if } 0 \leq s_1 \leq \frac{1}{2}, \\ g(2s_1 - 1, s_2, ..., s_m) & \text{if } \frac{1}{2} \leq s_1 \leq 1. \end{cases}$$

This sum is well-defined on homotopy classes, as one can check. This generalizes the operation on $\pi_1(X)$.

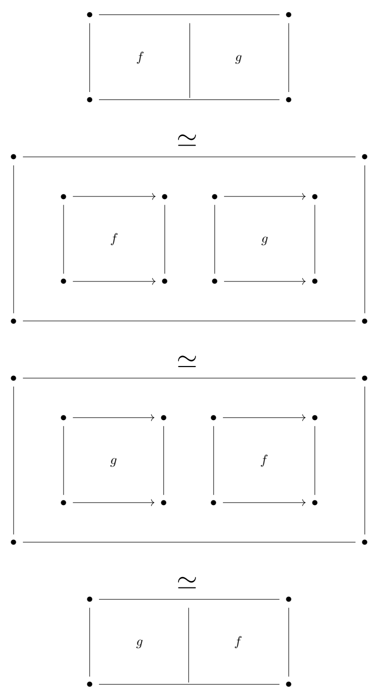
We can verify some properties related to the group structure of $\pi_n(X)$:

- The identity element is the (homotopy class of the) constant map.
- Inverses are $f(1 - s_1, s_2, ..., s_n)$.

Lemma 2.14.2. *If $f, g \in \pi_n(X)$, $n \geq 2$, then $f + g \simeq g + f$. Thus $\pi_n(X)$ is abelian, while $\pi_1(X)$ is not!*

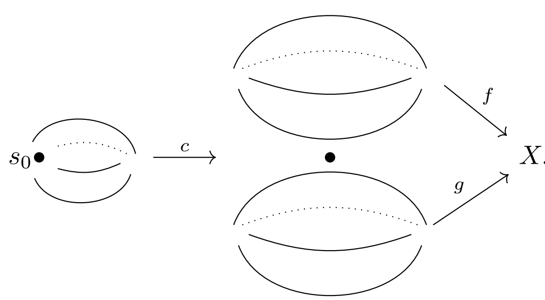
Proof. For this proof, we give a brief sketch. Consider the pictures describing how we manipulate the homotopy. In particular, we only need to modify the first two coordinates:

[22] Note that the x_0 is obviously a vacuous condition.
[23] E.g. what is the "identity path component?"



$\square$

We remark that a map $(I^n, \partial I^n) \to (X, x_0)$ is the same thing as a map $(S^n, s_0) \to (X, x_0)$. In this interpretation, $f + g$ can be viewed as the composition $(f \vee g)c$, where c collapses the equator. The following picture demonstrates:



Thus $\pi_n(X, x_0)$ is the homotopy classes of maps

$$\begin{tikzcd}
(I^n, \partial I^n) \arrow[rr] \arrow[dr, bend right] & & (X, x_0) \\
& (S^n, s_0) \arrow[ur, bend right] &
\end{tikzcd}$$

Proposition 2.14.3. *A map $f : (X, x_0) \to (Y, y_0)$ induces a homomorphism $f_* : \pi_n(X, x_0) \to \pi_n(Y, y_0)$. Moreover, if f is a homotopy equivalence, then f_* is an isomorphism.*

In other words, like π_1, everything is functorial.

Next, let $\gamma : I \to X$ and let $f : (I^n, \partial I^n) \to (X, x_0)$, where $\gamma(0) = x_0$ and $\gamma(1) = x_1$. Define $\beta_\gamma(f) : (I^n, \partial I^n) \to (X, x_1)$ by shrinking the domain of f and using γ on raidal paths. Note when $n = 1$, $\beta_\gamma f = \overline{\gamma} f \gamma$, so this is just the change of basepoint isomorphism for higher homotopy groups. Let's discuss a few of its properties:

- If $\gamma \simeq \gamma'$ and $f \simeq f'$, then $\beta_\gamma(f) \simeq \beta_{\gamma'}(f')$.
- $\beta_\gamma(f + g) \simeq \beta_\gamma(f) + \beta_\gamma(g)$.

Together, these properties tell us that we have a homomorphism $\beta_{[\gamma]} : \pi_n(X, x_0) \to \pi_n(X, x_1)$. We further remark that

- $\beta_{\gamma\eta}(f) \simeq \beta_\eta \beta_\gamma(f)$, and
- If c is a constant path, $\beta_c(f) \simeq f$.

Together, these properties tell us that we have an isomorphism where ${\beta_{[\gamma]}}^{-1} = \beta_{[\overline{\gamma}]}$.

Furthermore, we can consider the special case when γ is a loop; i.e., $\gamma(0) = \gamma(1)$. Then $\beta_{\gamma\eta}(f) \simeq \beta_\eta\beta_\gamma(f)$ implies that $\pi_1(X, x_0) \to \mathrm{Aut}(\pi_n(X, x_0))$, $[\gamma] \mapsto \beta_{[\gamma]}$ is a homomorphism. In other words, $\pi_1(X, x_0)$ acts on $\pi_n(X, x_0)$. Since $\pi_n(X, x_0)$ is abelian, we can think of $\pi_n(X, x_0)$ as a module over the group ring:

$$\mathbf{Z}[\pi_1(X, x_0)] = \left\{ \sum n_{[\gamma]}[\gamma] \mid \text{ finite sum } n_{[\gamma]} \in \mathbf{Z} \right\},$$

with module structure $\sum n_{[\gamma]}[\gamma] \cdot [f] = \sum n_{[\gamma]} \beta_{[\gamma]}([f])$.

Lemma 2.14.4. *A covering space projection $p : (\widetilde{X}, \widetilde{x_0}) \to (X, x_0)$ induces isomorphism on π_n for $n \geq 2$.*

In other words, covering spaces don't simplify maps of S^n, unlike $n = 1$.

Proof. Injectivity is the same as for $n = 1$; lift homotopies.

For surjectivity, S^n is simply connected for $n \geq 2$, so there always exists a lift:

$$\begin{tikzcd}
 & (\widetilde{X}, \widetilde{x_0}) \arrow[d, "p"] \\
(S^n, s_0) \arrow[r, "f"] \arrow[ur, dotted, bend left, "\widetilde{f}"] & (X, x_0)
\end{tikzcd}$$

Thus covering spaces don't simplify the picture and make π_n any easier to compute. $\square$

Corollary 2.14.5. *If the universal cover of X is contractible, then $\pi_n(X) = 0$ for $n \geq 2$.*

Example 2.14.6. $\pi_n(S^1) = 0$ for $n \geq 2$, since the universal cover is $\mathbf{R}$.

Example 2.14.7. $\pi_n(T^m) = \pi_m(S^1 \times ... \times S^1) = 0$ for $n \geq 2$, since the universal cover is $\mathbf{R}^m$.

Example 2.14.8. Let $X = S^1 \vee S^2$. Then $\pi_2(X) = \bigoplus_{\mathbf{Z}} \mathbf{Z}$; $\pi_2(X)$ is infinitely generated. The universal cover is Christmas lights; clearly a sphere can be mapped into every bulb. But, in the perspective of a module over a ring, $\pi_2(X) = \mathbf{Z}[\pi_1(X)] = \mathbf{Z}[t, t^{-1}]$, i.e., Laurent polynomials. See **Example 2.14.22**.

Proposition 2.14.9. *The following are equivalent:*

1. *For all $f : S^n \to X$, f is homotopic to a constant map; i.e., f is nulhomotopic (**Definition 1.51.4**).*
2. *For all $f : S^n \to X$, f extends to $F : D^{n+1} \to X$.*
3. *$\pi_n(X, x_0) = 0$ for all $x_0 \in X$.*

Lemma 2.14.10 (Extension Lemma). *Given a CW pair (X, A) and a map $f : A \to Y$ with Y path connected, there exists an extension $F : X \to Y$, provided that $\pi_{n-1}(Y) = 0$ for all n such that $X \setminus A$ has cells of dimension n.*

Proof. Assume inductively that F has been defined over the $(n-1)$-skeleton of X. We can extend F to the n-cell ${e^n}_\alpha$ if and only if the composition of the attaching map $\Phi_\alpha : \partial {D^n}_\alpha \to X^{n-1}$ with $F : X^{n-1} \to Y$ is nulhomotopic. This is equivalent to $F_*[\Phi_\alpha] = 0 \in \pi_{n-1}(Y)$. Thus, if $\pi_{n-1}(Y) = 0$, such an extension always exists. $\square$

Example 2.14.11. Suppose $A \subseteq X$ is a contractible subcomplex. Then $\pi_n(A) = 0$ for all n. Thus, the identity map on A, $id_A : A \to A$, extends to a retraction $r : X \to A$.

Proposition 2.14.12 (Whitehead's Theorem). *If X and Y are path connected CW-complexes and $f : X \to Y$ induces an isomorphism $f_* : \pi_n(X) \to \pi_n(Y)$ for all n, then f is a homotopy equivalence.*

Moreover, if f is the inclusion of a subcomplex $X \hookrightarrow Y$, then Y deformation retracts[24] *onto X.*

Corollary 2.14.13. *If $\pi_n(X) = 0$ for all n, then X deformation retracts onto a point; i.e., X is contractible.*

Beware, however, the following nonexample:

[24] Recall **Definition 1.58.6**.

Example 2.14.14. Let our spaces be $\mathbf{RP}^2$ and $S^2 \times \mathbf{RP}^\infty$. Then $\pi_1(\mathbf{RP}^2) = \mathbf{Z}_2 = \pi_1(S^2 \times \mathbf{RP}^\infty$. Furthermore, $\pi_n(\mathbf{RP}^2) = \pi_n(S^2 \times \mathbf{RP}^\infty)$ for all $n \geq 2$, as $\widetilde{\mathbf{RP}^2} = S^2$ and $\widetilde{S^2 \times \mathbf{RP}^\infty} = S^2 \times S^\infty$ are homotopy equivalent, as S^∞ is contractible.

However, $H^3(\mathbf{RP}^2; \mathbf{Z}_2) = 0$, while $H^3(S^2 \times \mathbf{RP}^\infty; \mathbf{Z}_2) = \bigoplus_{i+j=3} H^i(S^2; \mathbf{Z}_2) \otimes_{\mathbf{Z}_2} H^j(\mathbf{RP}^\infty; \mathbf{Z}_2) = \mathbf{Z}_2 \oplus \mathbf{Z}_2$, as $H^*(\mathbf{RP}^\infty; \mathbf{Z}_2) = \mathbf{Z}_2[\alpha]$ implies that $H^j(\mathbf{RP}^\infty; \mathbf{Z}_2) = \mathbf{Z}_2$. Thus $\mathbf{RP}^2$ and $S^2 \times \mathbf{RP}^\infty$ are not homotopy equivalent.

For **Whitehead's Theorem 2.14.12**, we must have the map $f : X \to Y$ to induce the isomorphism.

Now we develop the tools to compute $\pi_n(S^k)$.

Proposition 2.14.15 (Cellular Approximation Theorem). *Every map $f : X \to Y$ between CW-complexes is homotopic to a cellular map; i.e., $f(X^n) \subseteq Y^n$ (**Definition 2.2.10**).*

Corollary 2.14.16. *$\pi_n(S^k) = 0$ for $n < k$, since we can have a nonsurjective map.*

Definition 2.14.17. A space X is ***n*-connected** if $\pi_k(X) = 0$ for $k \leq n$.

Proposition 2.14.18 (Freudenthal Suspension Theorem). *The suspension map $\pi_i(X) \to \pi_{i+1}(SX)$ with X $(n-1)$-connected is an isomorphism for $i < 2n-1$ and a surjection for $i = 2n-1$.*

Corollary 2.14.19. *The map $\pi_n(S^n) \to \mathbf{Z}$ given by $[f] \mapsto \deg(f)$ is an isomorphism. In other words, $\pi_n(S^n) \cong \mathbf{Z}$.*

Proof. For $n = 1$, $\pi_1(S^1) \to \pi_2(S^2)$ is a surjection $(i = 1 = 2n-1)$.

For $n \geq 2$, $\pi_n(S^n) \to \pi_{n+1}(S^{n+1})$ is an isomorphism $(i < 2n-1)$. Thus, $\pi_n(S^n)$ is cyclic for all n.

Since $f \simeq g$ implies $\deg(f) = \deg(g)$ (**Lemma 2.8.58**), and for all $d \in \mathbf{Z}$, there exists $f : S^n \to S^n$ such that $\deg(f) = d$, $|\pi_n(S^n)| = \infty$. Thus, $\pi_n(S^n)$ is infinite cyclic, and is hence $\mathbf{Z}$.

To see that this is an isomorphism, since $\pi_1(S^1) \to \mathbf{Z}$, $[f] \mapsto \deg(f)$ is an isomorphism, so are $\pi_n(S^n) \to \mathbf{Z}$, as $\deg(f) = \deg(Sf)$. $\square$

We remark that we now know $\pi_k(S^n) = 0$ if $k < n$ and $\pi_n(S^n) = \mathbf{Z}$. However, higher homotopy groups don't behave so nicely, as we will soon see.

Proposition 2.14.20 (Hurewicz Theorem). *If X is $(n-1)$-connected where $n \geq 2$, then $\widetilde{H}_k(X) = 0$ for $k < n$ and $\pi_n(X) \cong H_n(X)$.*

Equivalently, if X is simply-connected (1-connected), then the first nonzero homotopy and homology groups are isomorphic and in the same dimension.

Corollary 2.14.21. *$\pi_n(S^n) \cong H_n(S^n) \cong \mathbf{Z}$, as S^n is $(n-1)$-connected.*

Hurewicz tells us it holds for $n \geq 2$, but we also already knew that this corollary held for $n = 1$.

Example 2.14.22. Let $X = S^1 \vee S^2$. Then the universal cover $\widetilde{X}$ is the Christmas lights, which is homotopy equivalent to $\bigvee_{\mathbf{Z}} S^2$; simply contract the line. We know that $\pi_2(X) = \pi_2(\widetilde{X})$ by **Lemma 2.14.4**, and $\widetilde{X}$ is contractible, so $\pi_2(\widetilde{X}) = H_2(\widetilde{X})$ by **Hurewicz 2.14.20**, and $H_2(\widetilde{X}) = \bigoplus_{\mathbf{Z}} \mathbf{Z} = \mathbf{Z}[t, t^{-1}]$, which acts on $\pi_1(X)$ by shifting the line.

If $\alpha \in \pi_1(X)$ is a generator, then $\alpha \cdot t^n = t^{n \pm 1}$.

Lemma 2.14.23. *Let X be path connected. Then the following are equivalent.*

1. *X is contractible.*

2. *$\pi_n(X) = 0$ for all n; i.e., X is n-connected for all n.*

3. *X is simply connected and $\widetilde{H_n}(X) = 0$ for all n; i.e., X is acyclic.*

Proof. 1. implies 3. is obvious. Contract X to a point, and we know that $\widetilde{H_n}(\text{point}) = 0$.

3. implies 2. follows from **Hurewicz 2.14.20**.

2. implies 1. follows from **Whitehead 2.14.12**. □

Definition 2.14.24. A **fibration** consists of

$$\begin{array}{ccc} F & \longrightarrow & E \\ & & \downarrow{\scriptstyle p} \\ & & B \end{array}$$

where F is the fiber, E is the total, and B is the base space, such that

1. for all $b \in B$, $p^{-1}(b)$ is homemorphic to F. We say that E is a Fs worth of B.

2. for all $b \in B$, there exists a neighborhood $U \subseteq B$ of b such that $p^{-1}(U) = U \times F$ and, via this homeomorphism, p is projection onto the first product. We say that B is locally a product, or locally trivial.

Example 2.14.25. The trivial bundle is

$$\begin{array}{ccc} F & \longrightarrow & B \times F \\ & & \downarrow \\ & & B \end{array}$$

Example 2.14.26. A covering space (**Definition 1.53.1**) is a fibration:

$$\begin{array}{ccc} F & \longrightarrow & \widetilde{X} \\ & & \downarrow{\scriptstyle p} \\ & & X \end{array}$$

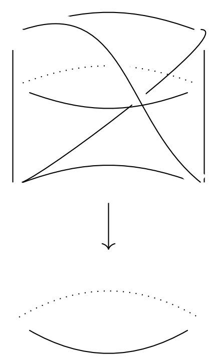
Here, F is a discrete space, with cardinality equal to the degree of the cover (**Definition 2.5.3**). That's the stack of records condition.

Example 2.14.27. Another fibration is

$$\begin{array}{ccc} I & \longrightarrow & \text{Möbius band} \\ & & \downarrow \\ & & S^1 \end{array}$$

i.e., we have

$I \longrightarrow$



Example 2.14.28. Yet another fibration is

$$\begin{array}{ccc} X & \longrightarrow & M_f \\ & & \downarrow \\ & & S^1 \end{array}$$

where, if $f : X \to X$ is a homeomorphism, then $M_f = {X \times [0,1]} \big/ {(x,1) \sim (f(x),0)}$ and the projection map is the t coordinate mod 1.

In fact, the Möbius band is just M_f where $f : [-1,1] \to [-1,1]$, $x \mapsto -x$.

Example 2.14.29. A further fibration is

$$\begin{array}{ccc} \mathbf{Z}_2 & \longrightarrow & S^n \\ & & \downarrow{\scriptstyle p} \\ & & \mathbf{RP}^n \end{array}$$

which we can complexify to get

$$\begin{array}{ccc} S^1 & \longrightarrow & S^{2n+1} \\ & & \downarrow{\scriptstyle p} \\ & & \mathbf{CP}^n \end{array}$$

where we recall that

$$\mathbf{CP}^n = \left\{(z_0, ..., z_n) \in \mathbf{C}^{n+1} \mid \sum |z_i|^2 = 1\right\} \Big/ (z_0, ..., z_n) \sim \lambda(w_0, ..., w_n),$$

where $\lambda \in S^1 \subseteq \mathbf{C}$; i.e., $|\lambda| = 1$.

Example 2.14.30. In the special case of **Example 2.14.29** above when $n = 1$, $\mathbf{CP}^1 = S^2$ and we have

$$\begin{tikzcd} S^1 \arrow[r] & S^3 \arrow[d, "p"] \\ & S^2 \end{tikzcd}$$

This is called the Hopf bundle.

We make a few remarks about $\mathbf{CP}^1$.

> Let $\mathbf{CP}^1 = U_0 \cup U_1$ where $U_0 = \{[(z_0, z_1)] \mid z_0 \neq 0\}$; any such equivalence class can be represented by $\left(1, \frac{z_1}{z_0}\right)$, and $U_1 = \{[(z_0, z_1)] \mid z_1 \neq 0\}$; any such equivalence class can be represented by $\left(\frac{z_0}{z_1}, 1\right)$. Each U_i is homeomorphic to $\mathbf{C}$. Then $\mathbf{CP}^1 = U_0 \cup [(0,1)] = [(1,0)] \cup U_1$; i.e., $\mathbf{CP}^1 = \mathbf{C} \cup \{\infty\}$.

Note that $S^3 \subseteq \mathbf{C}^2$, so the map $p(z_0, z_1) = \frac{z_0}{z_1} \in \mathbf{C} \cup \{\infty\}$. In polar coordinates, $p\left(r_0 e^{i\theta_0}, r_1 e^{i\theta_1}\right) = \frac{r_0}{r_1} e^{i(\theta_0 - \theta_1)}$, where ${r_0}^2 + {r_1}^2 = 1$.

For a fixed ratio $\rho = \frac{r_0}{r_1}$, the angles θ_0 and θ_1 can vary independently over S^1, so this gives a "torus" $T_\rho = \{(r_0 e^{i\theta_0}, r_1 e^{i\theta_1})\} \subseteq S^3$.

Letting ρ vary, we can fill S^3 with tori, including the degenerate cases $T_0 = \{(0, e^{i\theta_1})\}$ and $T_\infty = \{(e^{i\theta_0}, 0)\}$.

We can use stereographic projection from $(1,0) \in \mathbf{C}^2$ to visualize this in $\mathbf{R}^3$, since $S^3 \setminus \{(1,0)\} \cong \mathbf{R}^3$. Then T_∞ corresponds to the z-axis, and T_0 is the unit circle in the xy-plane.

The fibration maps:

$$\begin{aligned} T_0 &\mapsto \text{ origin} \\ T_1 &\mapsto \text{ unit circle in } \mathbf{C} \\ T_\infty &\mapsto \text{ infinity} \\ T_r &\mapsto \text{ circle of radius } r. \end{aligned}$$

Each torus T_ρ is foliated by circles with $\theta_0 - \theta_1$ for a fixed θ.

Given a few more minutes, this will show that $\pi_3(S^2) \cong \mathbf{Z}$; the Hopf map generates the third homotopy group.

3 Index

3.1 Definitions

D

E

F

3.2 Appendix: Nice things to know, which may be fleshed out more in the future

$$\begin{tikzcd}
\text{Metrizable} \arrow[r, Rightarrow] & \text{Normal} \arrow[r, Rightarrow] & \text{Regular} \arrow[r, Rightarrow] \arrow[ll, Rightarrow, bend right, "\text{second countable}"'] & \text{Hausdorff} \arrow[ll, Rightarrow, bend left, "\text{compact}"]
\end{tikzcd}$$