Constructing Projective Resolutions of Modules

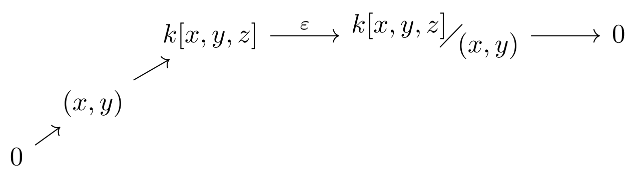
Let M be an R-module. A **projective (resp. free) resolution** of M is a complex $P_\bullet$ with P_i all projective (resp. free) modules and $P_i = 0$ for $i < 0$, together with a map $\varepsilon : P_0 \to M$ so that the augmented complex

$$\cdots \to P_2 \to P_1 \to P_0 \xrightarrow{\varepsilon} M \to 0$$

is exact. It is known (Lemma 2.2.5) that every R-module has a projective resolution; more generally, an abelian category with enough projectives always has projective resolutions.

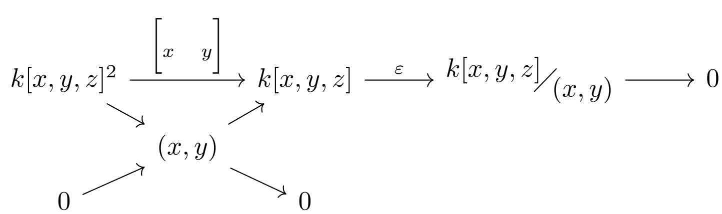
Exercise 1 Let k be a field. Consider the ring of polynomials $R = k[x, y, z]$. Compute a projective resolution for $M = {}^{k[x,y,z]}\!\big/_{(x,y)}$.

We build a free resolution, hence projective. See that $k[x, y, z]$ surjects onto ${}^{k[x,y,z]}\!\big/_{(x,y)}$ via $f \overset{\varepsilon}{\mapsto} [f]$. It has kernel (x, y).



There exists a surjection $k[x, y, z]^2 \to (x, y)$, namely $(f, g) \mapsto fx + gy$. As a result, we get a map $d_1 : k[x, y, z]^2 \to k[x, y, z]$ by composition, and note that since $k[x, y, z]^2 \to (x, y)$ is surjective, $\operatorname{im}(d_1) = \operatorname{im}((x, y) \to k[x, y, z])$. We may write d_1 as the matrix $\begin{bmatrix} x & y \end{bmatrix}$; then

$$d_1 \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix} = fx + gy.$$



The kernel of $\begin{bmatrix} x & y \end{bmatrix}$ is generated by $\begin{bmatrix} -y \\ x \end{bmatrix}$. By rank-nullity, since $k[x, y, z]^2$ is rank 2 and the image of d_1 is rank 1, we have

$$0 \longrightarrow k[x,y,z] \xrightarrow{\begin{bmatrix} -y \\ x \end{bmatrix}} k[x,y,z]^2 \xrightarrow{\begin{bmatrix} x & y \end{bmatrix}} k[x,y,z] \xrightarrow{\varepsilon} {}^{k[x,y,z]}\!\big/_{(x,y)} \longrightarrow 0.$$

One can use the program Macaulay2 at http://habanero.math.cornell.edu:3690/ to compute free resolutions of modules over polynomial rings. To demonstrate how to use it, we compute the previous exercise in Macaulay2. The commands are as follows.

```
i1 :  R=QQ[x,y,z]
i2 :  I=ideal(x,y)
i3 :  M=module R/I
i4 :  rs=res M
i5 :  rs.dd
```

Exercise 2 Compute a projective resolution for $M = {}^{k[x,y,z]}\!\big/_{(xy,xz)}$.

As always, we have

$$\begin{array}{ccccccc} & & k[x,y,z] & \xrightarrow{\varepsilon} & {}^{k[x,y,z]}\!\big/_{(xy,xz)} & \longrightarrow & 0 \\ & \nearrow & & & & & \\ (xy,xz) & & & & & & \\ \nearrow & & & & & & \\ 0 & & & & & & \end{array}$$

And as there are two generators of the ideal, we have the map $d_1 : k[x,y,z]^2 \to k[x,y,z]$ given by $d_1 = \begin{bmatrix} xy & xz \end{bmatrix}$. The kernel of d_1 is generated by $\begin{bmatrix} -z \\ y \end{bmatrix}$, since

$$\begin{bmatrix} xy & xz \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix} = xyf + xzg = x(yf + zg),$$

and this is zero precisely when described. Thus the free resolution is

$$0 \longrightarrow k[x,y,z] \xrightarrow{\begin{bmatrix} -z \\ y \end{bmatrix}} k[x,y,z]^2 \xrightarrow{\begin{bmatrix} xy & xz \end{bmatrix}} k[x,y,z] \xrightarrow{\varepsilon} {}^{k[x,y,z]}\!\big/_{(xy,xz)} \longrightarrow 0.$$

Checking our work against Macaulay2 (note to write `I=ideal(x*y,x*z)`!) confirms our work is correct.

Exercise 3 Compute a projective resolution for $M = {}^{k[x,y,z]}\!\big/_{(xy,xz,x^3)}$.

We have

$$k[x,y,z]^3 \xrightarrow{[\,xy\ xz\ x^3\,]} k[x,y,z] \xrightarrow{\varepsilon} {}^{k[x,y,z]}\!\big/_{(xy,xz,x^3)} \longrightarrow 0.$$

We need to explore the kernel of $\left[\begin{smallmatrix} xy & xz & x^3 \end{smallmatrix}\right]$. Observe that

$$\left[\begin{smallmatrix} xy & xz & x^3 \end{smallmatrix}\right]\left[\begin{smallmatrix} f \\ g \\ h \end{smallmatrix}\right] = xyf + xzg + x^3h = x(yf + zg + x^2h).$$

See that the elements $\left[\begin{smallmatrix} -z \\ y \\ 0 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} -x^2 \\ 0 \\ y \end{smallmatrix}\right]$, and $\left[\begin{smallmatrix} 0 \\ -x^2 \\ z \end{smallmatrix}\right]$ are elements of the kernel. Thus, we have

$$k[x,y,z]^3 \xrightarrow{\left[\begin{smallmatrix} -z & -x^2 & 0 \\ y & 0 & -x^2 \\ 0 & y & z \end{smallmatrix}\right]} k[x,y,z]^3 \xrightarrow{\left[\begin{smallmatrix} xy & xz & x^3 \end{smallmatrix}\right]} k[x,y,z] \xrightarrow{\varepsilon} k[x,y,z]/(xy, xz, x^3) \longrightarrow 0.$$

In considering the kernel of $\left[\begin{smallmatrix} -z & -x^2 & 0 \\ y & 0 & -x^2 \\ 0 & y & z \end{smallmatrix}\right]$, we can see that

$$\left[\begin{smallmatrix} -z & -x^2 & 0 \\ y & 0 & -x^2 \\ 0 & y & z \end{smallmatrix}\right]\left[\begin{smallmatrix} f \\ g \\ h \end{smallmatrix}\right] = \left[\begin{smallmatrix} -zf-x^2g \\ yf-x^2h \\ yg+zh \end{smallmatrix}\right].$$

It is sufficient that $f = x^2$, $g = -z$, and $h = y$, for then $\left[\begin{smallmatrix} -zf-x^2g \\ yf-x^2h \\ yg+zh \end{smallmatrix}\right] = \left[\begin{smallmatrix} -zx^2-x^2(-z) \\ yx^2-x^2y \\ y(-z)+zy \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 \\ 0 \\ 0 \end{smallmatrix}\right]$. Therefore, our final answer is

$$\begin{array}{c} 0 \\ \big\downarrow \\ k[x,y,z] \\ \big\downarrow \left[\begin{smallmatrix} x^2 \\ -z \\ y \end{smallmatrix}\right] \\ k[x,y,z]^3 \\ \big\downarrow \left[\begin{smallmatrix} -z & -x^2 & 0 \\ y & 0 & -x^2 \\ 0 & y & z \end{smallmatrix}\right] \\ k[x,y,z]^3 \\ \big\downarrow \left[\begin{smallmatrix} xy & xz & x^3 \end{smallmatrix}\right] \\ k[x,y,z] \\ \big\downarrow \varepsilon \\ k[x,y,z]/(xy, xz, x^3) \\ \big\downarrow \\ 0. \end{array}$$

Exercise 4 Consider the ring $S = k[x,y,z,u,v,w]$. Compute a projective resolution for $M = k[x,y,z,u,v,w] \big/ \left(\left| \begin{smallmatrix} x & y \\ u & v \end{smallmatrix} \right|, \left| \begin{smallmatrix} y & z \\ v & w \end{smallmatrix} \right|, \left| \begin{smallmatrix} x & z \\ u & w \end{smallmatrix} \right| \right)$.

Write $\Delta_1 = \left| \begin{smallmatrix} x & y \\ u & v \end{smallmatrix} \right|$, $\Delta_2 = \left| \begin{smallmatrix} y & z \\ v & w \end{smallmatrix} \right|$, and $\Delta_3 = \left| \begin{smallmatrix} x & z \\ u & w \end{smallmatrix} \right|$. The first steps proceed as normal.

$$S^3 \xrightarrow{[\,\Delta_1\ \Delta_2\ \Delta_3\,]} S \xrightarrow{\varepsilon} M \longrightarrow 0.$$

Now, as before, see that

$$[\,\Delta_1\ \Delta_2\ \Delta_3\,] \begin{bmatrix} f \\ g \\ h \end{bmatrix} = \Delta_1 f + \Delta_2 g + \Delta_3 h,$$

and that the elements $\begin{bmatrix} -\Delta_2 \\ \Delta_1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} -\Delta_3 \\ 0 \\ \Delta_1 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ -\Delta_3 \\ \Delta_2 \end{bmatrix}$ must be in the kernel. However, by comparing ranks, we see that the next free module must be of rank strictly less than 3, so there must be a way to express the kernel in fewer relations. Indeed, observe that the matrices

$$\begin{bmatrix} x & y & z \\ x & y & z \\ u & v & w \end{bmatrix} \text{ and } \begin{bmatrix} u & v & w \\ x & y & z \\ u & v & w \end{bmatrix}$$

must be singular, so computing their determinants by expanding along the top rows, we have $x\Delta_2 - y\Delta_3 + z\Delta_1 = 0$ and $u\Delta_2 - v\Delta_3 + w\Delta_1 = 0$. These now describe our kernel, and we have

$$0 \longrightarrow S^2 \xrightarrow{\begin{bmatrix} z & w \\ x & u \\ -y & -v \end{bmatrix}} S^3 \xrightarrow{[\,\Delta_1\ \Delta_2\ \Delta_3\,]} S \xrightarrow{\varepsilon} M \longrightarrow 0.$$

Over a polynomial ring, these free resolutions will always have finite length. It is worthwhile to check that the following will be an infinite free resolution. Let k be a field, let $R = k[x,y]$ and consider the ring $S = R\big/(x,y)$. The free resolution of the S-module $M = R\big/(x)$ is

$$\cdots \xrightarrow{[\,y\,]} S \xrightarrow{[\,x\,]} S \xrightarrow{[\,y\,]} S \xrightarrow{[\,x\,]} S \xrightarrow{\varepsilon} M \longrightarrow 0.$$